Local Constant Approximation for Dominating Set on Graphs Excluding Large Minors

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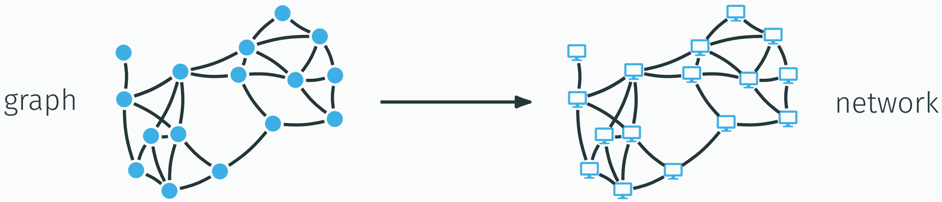
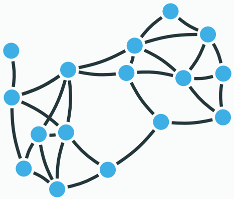
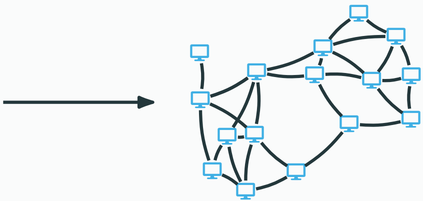
The LOCAL model



The LOCAL model

graph





network

vertices

+

edges

computers

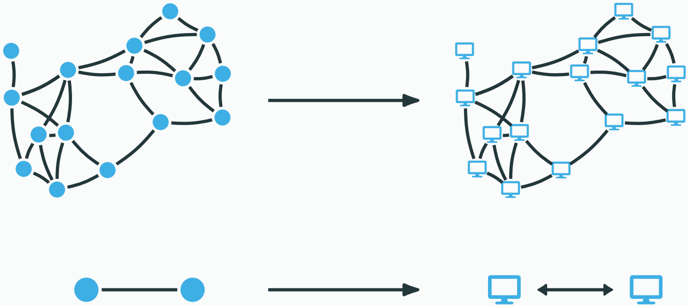
+

connections

UID UID

The LOCAL model

graph



network

vertices + edges

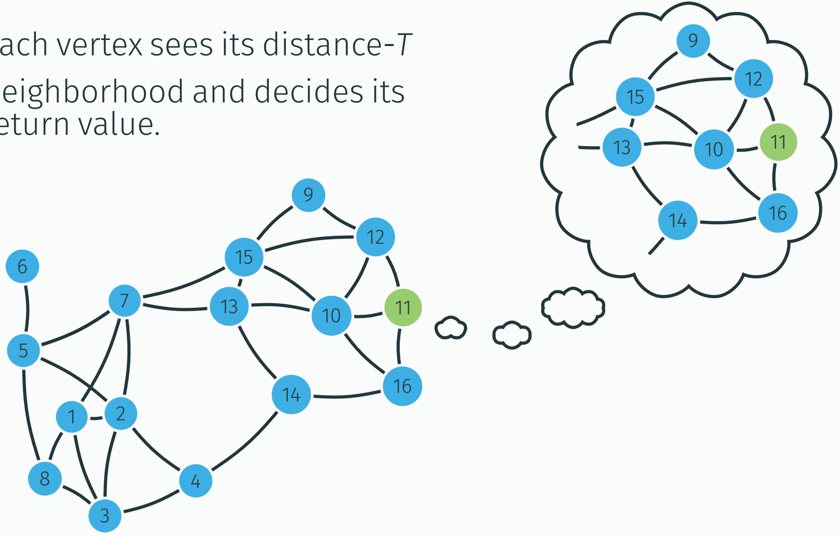
computers + connections

UID UID

The network is also the input graph!

Equivalence with number of rounds T

Each vertex sees its distance-T neighborhood and decides its return value.



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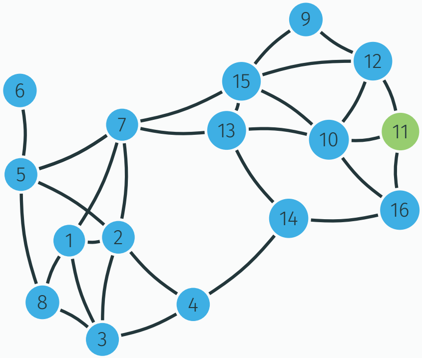
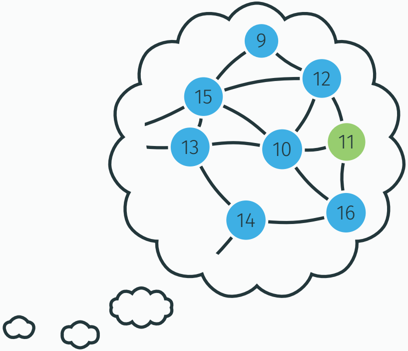
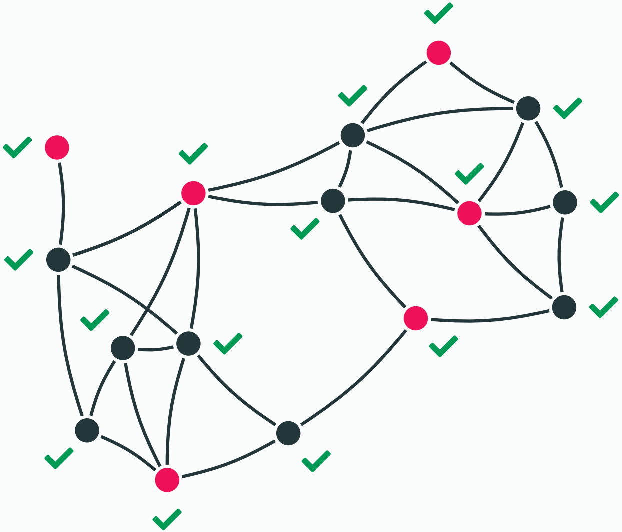




$$\text{Algo} = \mathcal{A} : \begin{matrix}\text{distance-T}\\ \text{neighborhood}\end{matrix} \mapsto \begin{matrix}\text{local}\\ \text{return value}\end{matrix}$$

An example: Minimum Dominating Set



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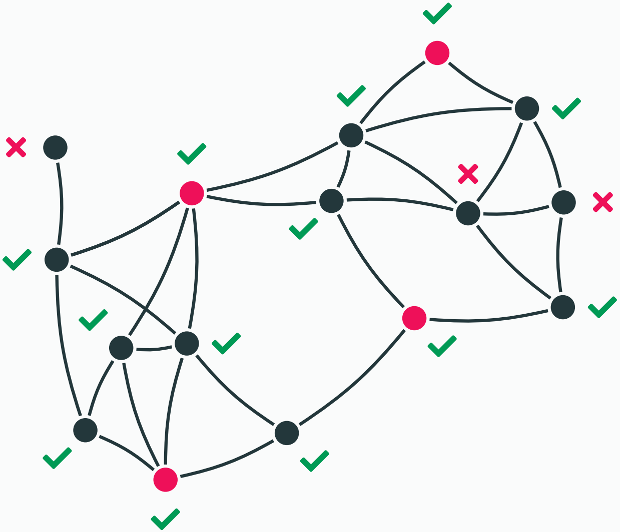

Is all hope lost?

Theorem (Kuhn, Moscibroda, and Wattenhofer, 2016)

It is ***impossible*** *to approximate* MINIMUM DOMINATING SET *with a constant number of rounds and constant approximation ratio on* ***general graphs****.*

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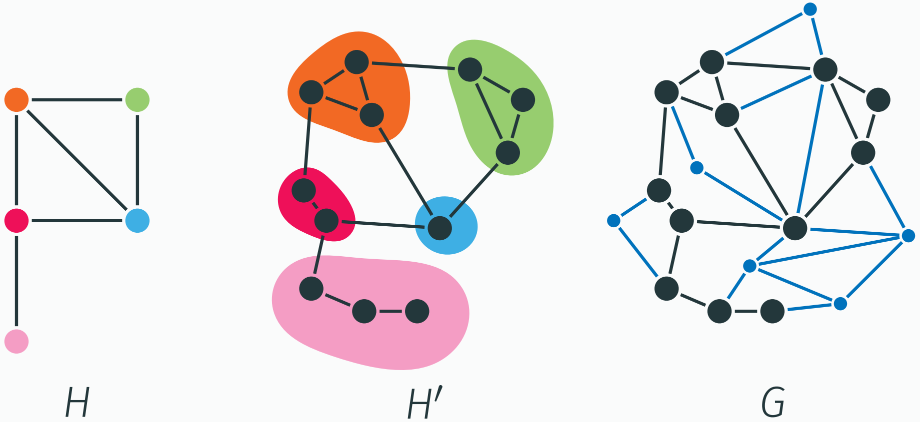
Restricting the graph class!



H is a minor of G

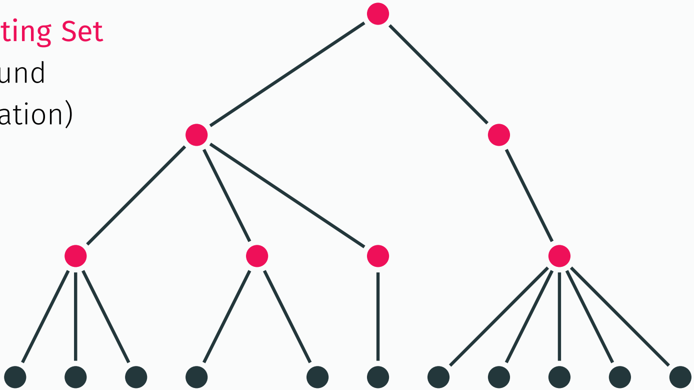
State of the art for MDS with $\mathcal{O}(1)$ LOCAL rounds

MINOR-FREE GRAPHS	APPROX. RATIO lower	APPROX. RATIO upper	#rounds
trees (K_3)	3[2]	3[2]	1
outerplanar (K_4, $K_{2,3}$)	5[3]	5[3]	2
planar (K_5, $K_{3,3}$)	7[1]	$11 + \varepsilon$[4]	$\mathcal{O}_\varepsilon(1)$
$K_{2,t}$-minor-free	5[3]	**$2t - 1$**	**3**
	5[3]	**50**	**$\mathcal{O}_t(1)$**
$K_{3,t}$-minor-free	7[1]	$(2 + \varepsilon) \cdot (t + 4)$[4]	$\mathcal{O}_{\varepsilon,t}(1)$
$K_{s,t}$-minor-free	7[1]	$t^{\mathcal{O}(st\sqrt{\log s})}$ [4]	$\mathcal{O}_t(1)$

[1] M. Hilke, C. Lenzen, and J. Suomela. Brief announcement: local approximability of minimum dominating set on planar graphs. PODC 2014.
[2] Folklore
[3] M. Bonamy, L. Cook, C. Groenland, and A. Wesolek. A tight local algorithm for the minimum dominating set problem in outerplanar graphs. DISC 2021.
[4] O. Heydt, S. Kublenz, P. Ossona de Mendez, S. Siebertz, and A. Vigny. Distributed domination on sparse graph classes. European Journal of Combinatorics, 2025.

Example 1: trees

- Dominating Set (one-round computation)



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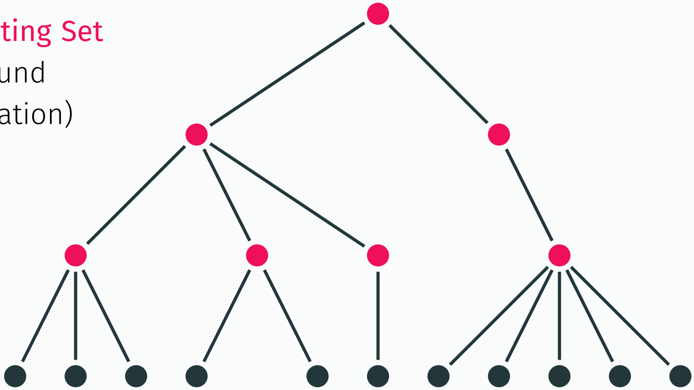
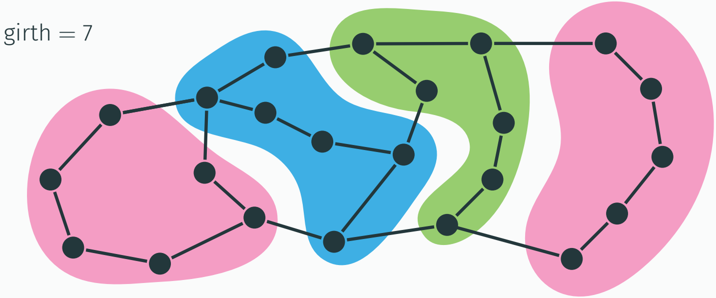


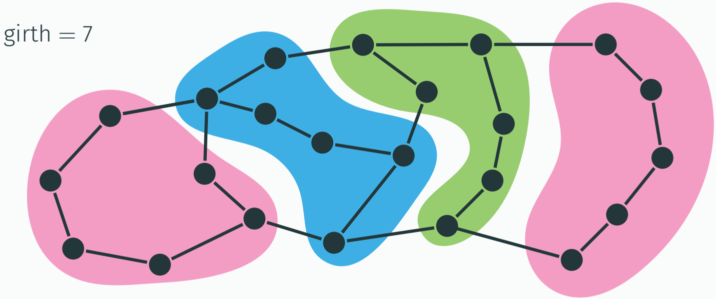
Theorem (folklore)

$|\{v \in V(T) \mid \deg(v) \geq 2\}| \leq 3 \cdot \mathrm{MDS}(T)$



Reusing the algorithm of trees?

Example 2: high girth graphs

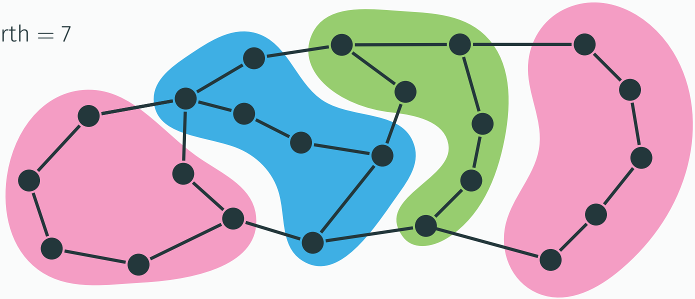


Reusing the algorithm of trees?

- Every vertex is in a bag ($\implies$ covering)

Example 2: high girth graphs

girth $= 7$



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- Diameter $\leq$ girth$/2$ ($\implies$ to see a tree)

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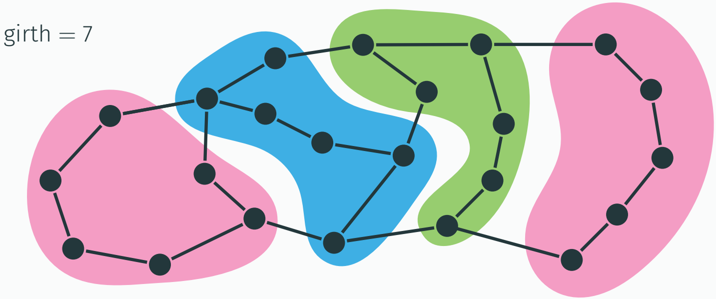

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- Spacing bags of same color ($\implies$ no overcounting for fixed color)

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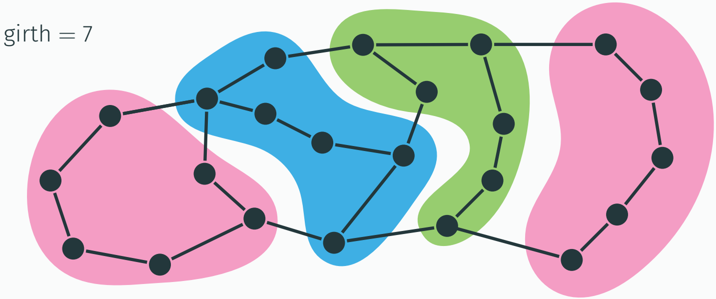

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- Diameter $\leq$ girth/2 ($\Longrightarrow$ to see a tree)
- Spacing bags of same color ($\Longrightarrow$ no overcounting for fixed color)
- Few colors ($\Longrightarrow$ to limit overcounting)

Asymptotic dimension

Asymptotic dimension of $\mathcal{C}$ is d if $\exists f : \mathbb{N} \to \mathbb{N}, \forall G \in \mathcal{C}, \forall r \in \mathbb{N}, \exists B_1, B_2, \dots \subseteq V(G)$, such that

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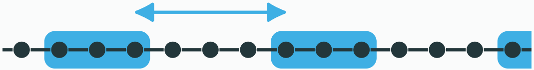

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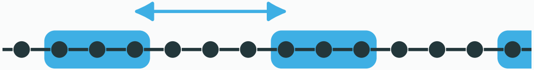
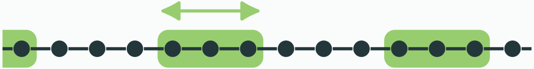
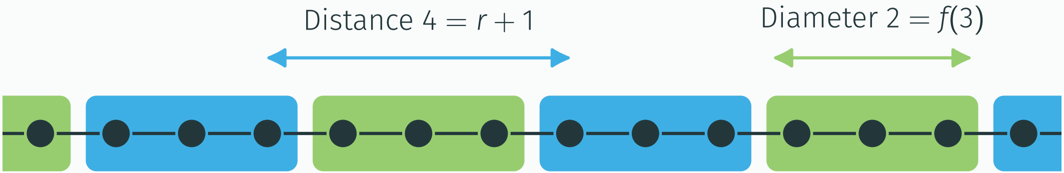


- **Boundedness:** $\forall i, \mathsf{diam}_G(B_i) \leq f(r)$



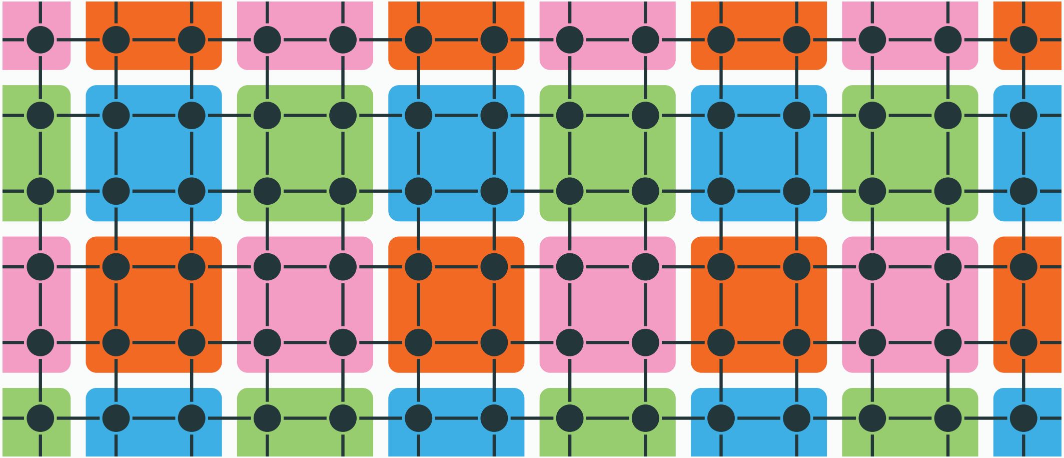
Example 1: the path ($r = 3$)



Dimension $= 1$ (2 colors)

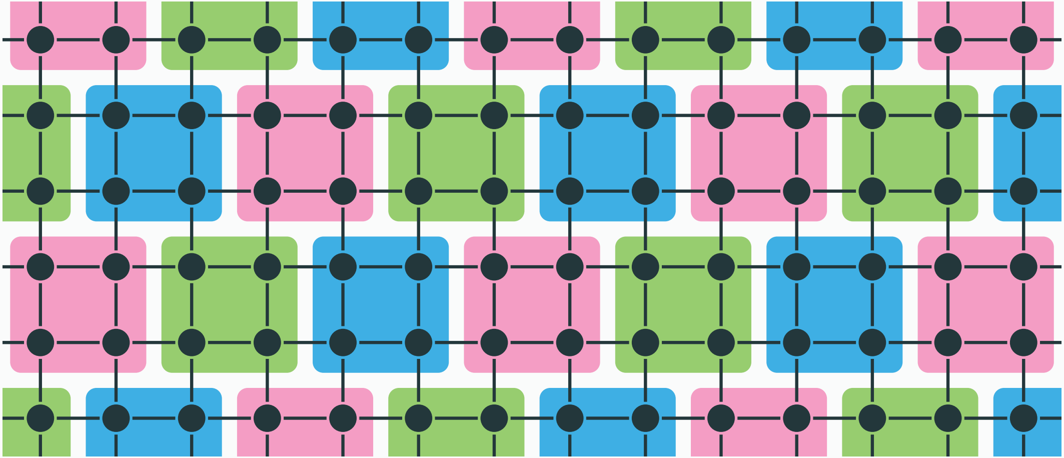
with $f(r) = r - 2$

Example 2: the grid – attempt 1 ($r = 2$)



Dimension ≤ 3

Example 2: the grid – attempt 2 ($r = 2$)



Dimension $= 2$!

Asymptotic dimension and graph minors

Theorem (Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, Scott, 2020)

Every class excluding a fixed minor has asymptotic dimension ≤ 2.

Application: distributed algorithms

How to use graph theory in distributed algorithms?

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Global concept

↓

Local concept

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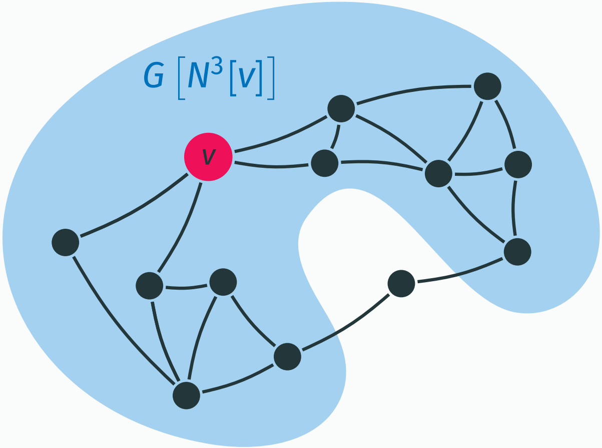
Local concept

Definition

v is a *r-local cutvertex* if v is a cutvertex of $G\left[N^r[v]\right]$.



Main theorem

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*On $K_{2,t}$-minor-free graphs, there exists a constant-approximation (where the constant is **independent of t**) of MINIMUM DOMINATING SET in the LOCAL model, in $f(t)$ rounds.*

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Previous bound on H-minor-free graphs had $\Omega(|V(H)|)$ in $g(H)$ rounds (Heydt, Kublenz, Ossona de Mendez, Siebertz, Vigny 2022).

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Asymptotic dimension only used in the analysis!

The algorithm

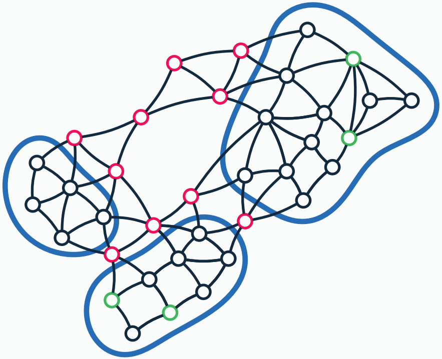
$$Y_t(G) = \bigcup\{18t\text{-local cuts of size } \leq 2\} \setminus \{\text{non-interesting vertices}\}$$
$$\text{Algorithm} = Y_t(G) \cup [\text{brute-force on } G - Y_t(G)]$$

Lemma 1

$|Y_t(G)| = \mathcal{O}(d) \cdot \mathrm{MDS}(G)$.

Lemma 2

If G is $K_{2,t}$-minor-free, every connected component of $G - Y_t(G)$ has diameter $\mathcal{O}_t(1)$.



Conclusion and perspectives

Follow-up works:

- H-minor-free graphs admit a 50-approximation if $\mathsf{pathwidth}(H) = 2$.
- Constant factor approximations in locally-nice graphs, e.g. bounded genus.
- Transform algorithms from a class $\mathcal{C}$ to a locally-$\mathcal{C}$ class.

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☻ Thanks! ☻