Locally finding small dominating sets in $K_{2,t}$-minor-free graphs

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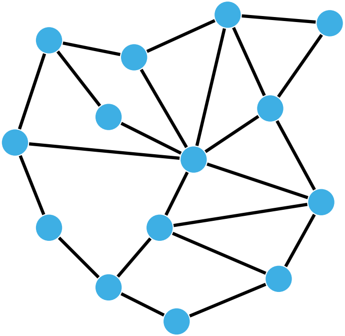
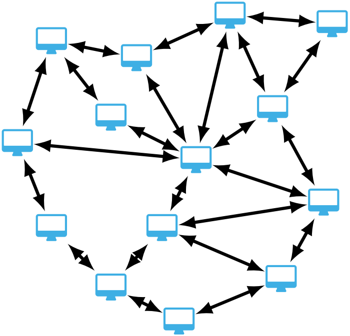
Distributed algorithms

Centralized



Distributed algorithm



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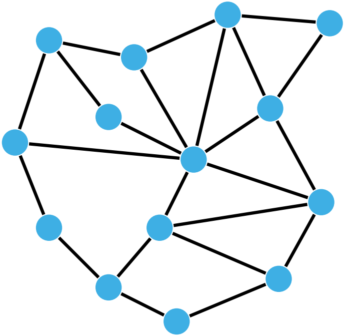

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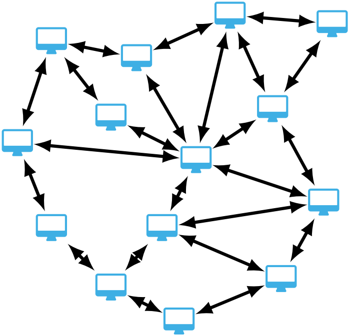

Focused on communication

The $\mathcal{LOCAL}$ model, motivation

- Real world fact:

transfer 1 bit in a local network

$$\Longleftrightarrow$$

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- Real world fact:

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do 1 billion arithmetic operations on a computer
(both take ≈ 0.5 ms)

- Therefore, each vertex/computer has an **infinite** computing power
- Difficulty of a problem = number of *rounds* required to solve it.

The $\mathcal{LOCAL}$ model, definition

Every vertex runs the same algorithm and has independent memory

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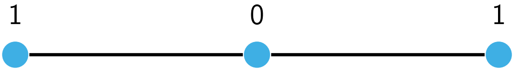
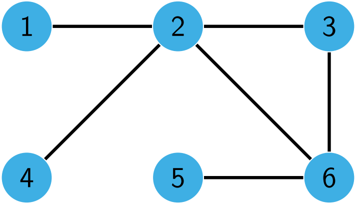


Return value of the algorithm: $\{v \in V(G)) \mid v \text{ returns } 1\}$

Identifiers and information



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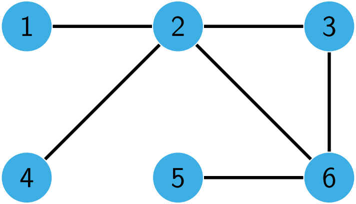

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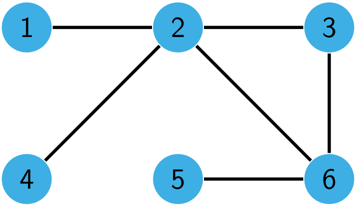

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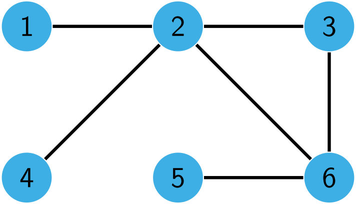

- Unique identifier per vertex
- No shared memory: every vertex only knows its ID + the info given by its neighbors.
- Only way to get info: communicate

Equivalent formulation of the $\mathcal{LOCAL}$ model

r-round algorithm $\iff$ no communication but every vertex v knows $N_r[v]$ and the corresponding IDs.

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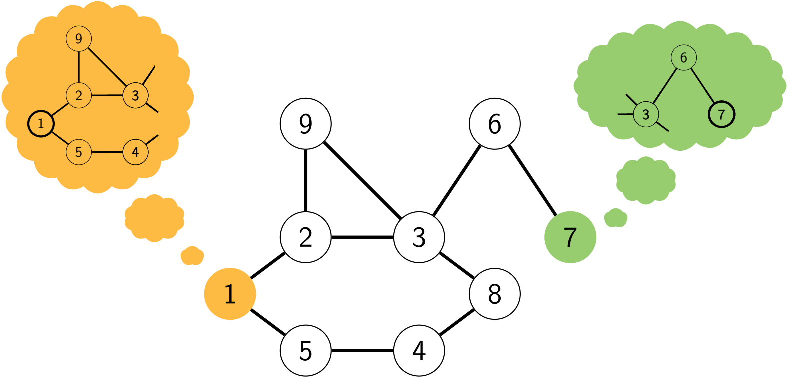
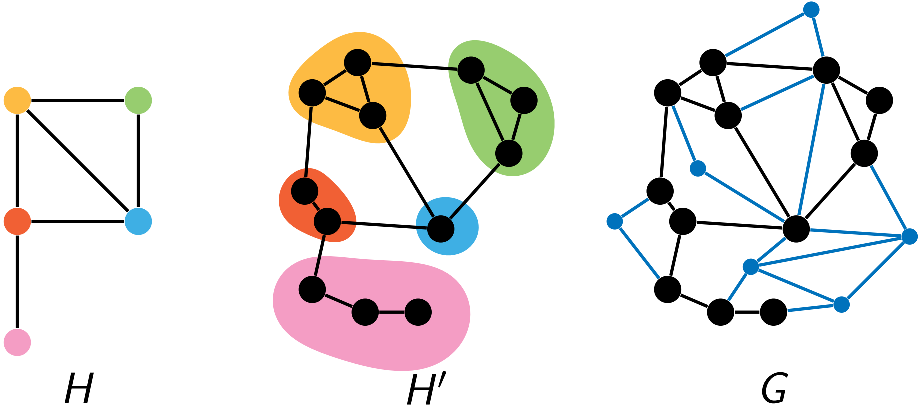
After 2 rounds:

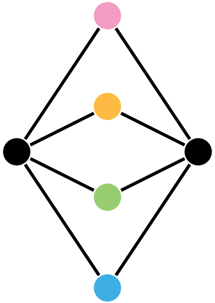


Graph minors

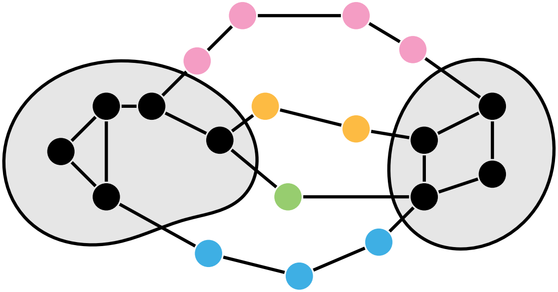


H is a minor of G

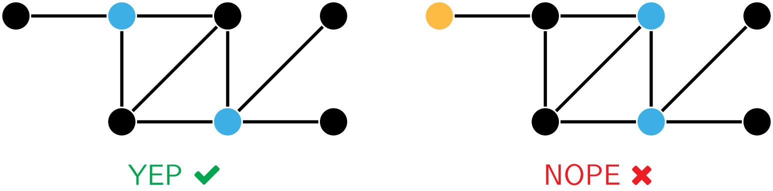
$K_{2,4}$



Instance of a $K_{2,4}$ minor in G



Minimum Dominating Set



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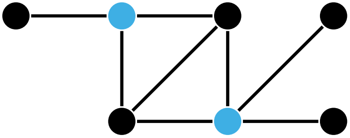

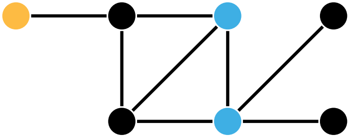

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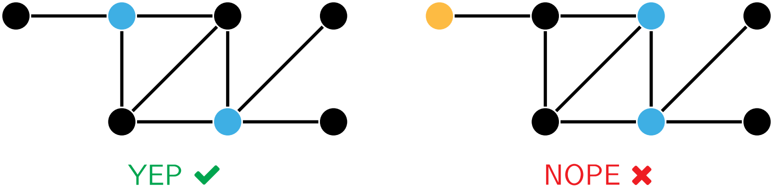

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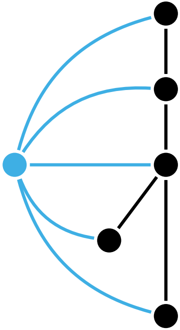
- In the centralized model: NP-complete (Karp) and hard to approximate.
- But hard in the centralized model $\not\Leftrightarrow$ hard in the $\mathcal{LOCAL}$ model.

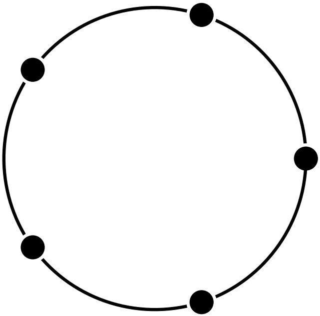
Differences in complexities between $\mathcal{LOCAL}$ and centralized

Maximum Independent Set
when ∃ universal vertex



Easy in $\mathcal{LOCAL}$
Hard in centralized

Detecting Cycles



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- General graphs
  - No constant factor approximation (Kuhn, Moscibroda and Wattenhofer 2016)

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- Outerplanar graphs
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- $K_{2,t}$-minor-free graphs
  - $(2t-1)$-approximation
  - Generalizes the outerplanar result

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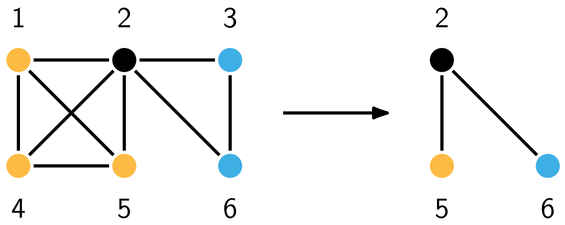

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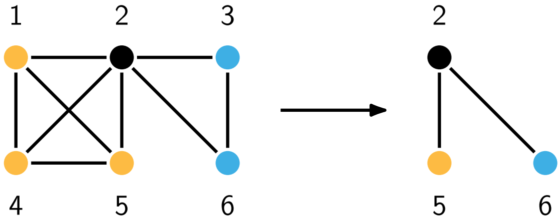

- Return $D_2 = \{v \in V(G) | \neg\exists u \in V(G - v), N[v] \subseteq N[u]\}$

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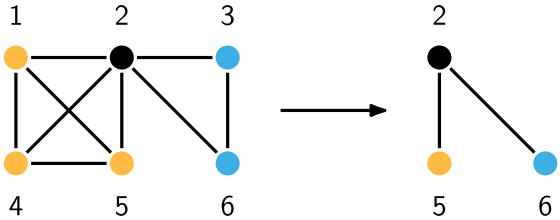

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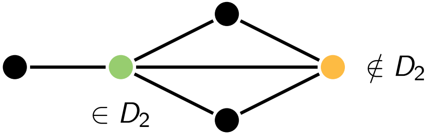

Approximation factor

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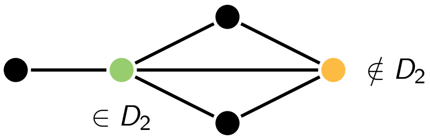


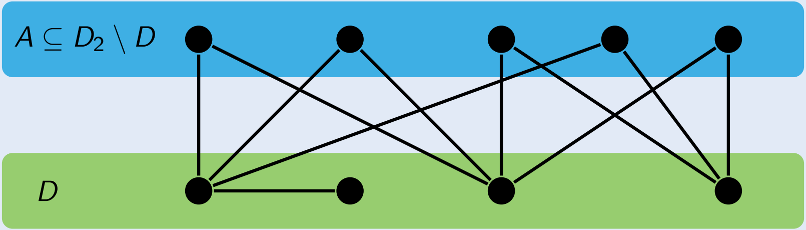
Theorem

Let D a MDS of G. If G is $K_{2,t}$-minor-free, then $|D_2| \leq (2t-1)|D|$.

Part 1: approximation factor

Lemma

Let D a MDS of G. Then $\exists H$ minor of G of the form:



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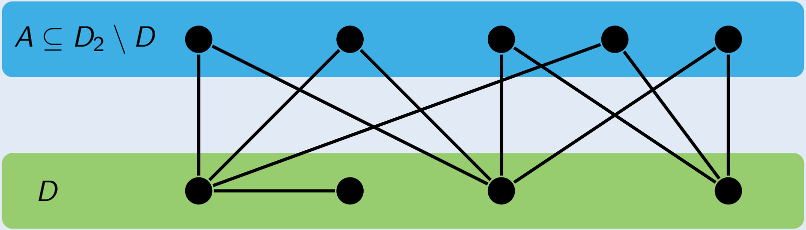

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$$\forall a \in A, |N(a) \cap D| \geq 2$$

Proof part 1: approximation factor

Lemma

Let $D = \{d_1, d_2, \ldots, d_k\}$ *a MDS of* G*. Then there exists* H *minor of* G *s.t.:*

- $V(H) = A \sqcup D$ *and* $A \subseteq D_2 \setminus D$
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Proof:

- Contract the branch sets $b_i = N[d_i] \setminus (D_2 \setminus D \cup \bigcup_{j<i} N[d_i] \cup \{d_{i+1}, \ldots, d_k\})$

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- Contract some edges so that every vertex left in $D_2 \setminus D$ has 2 neighbors in D

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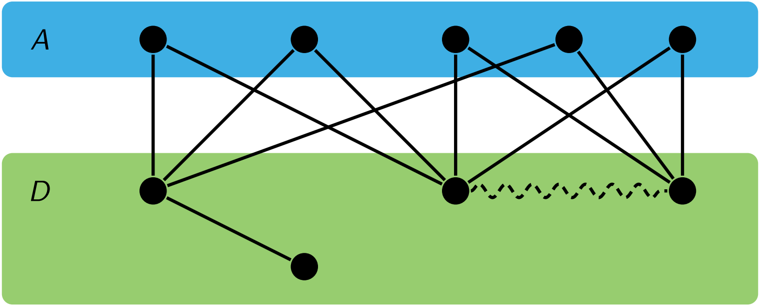

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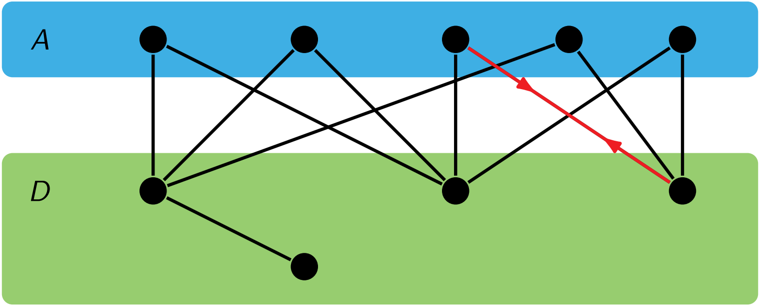

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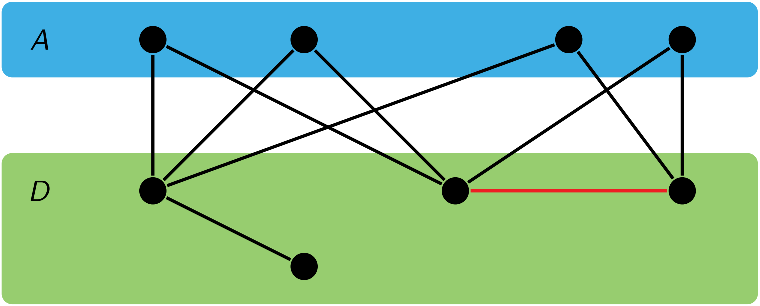

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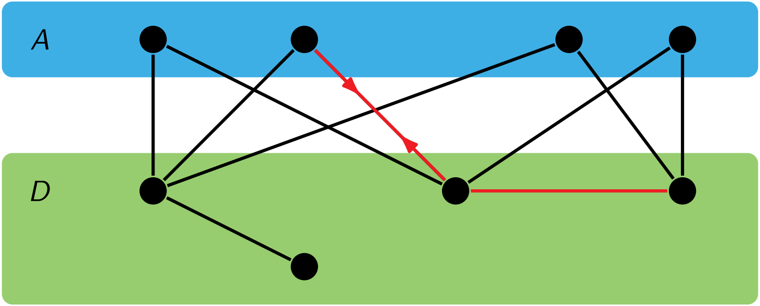

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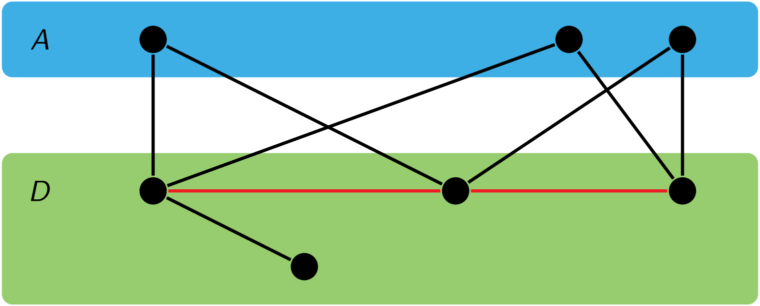

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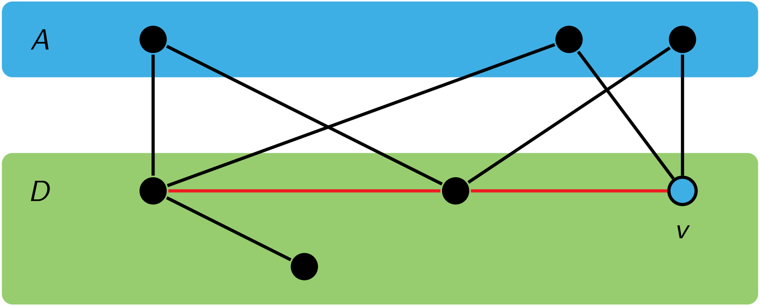

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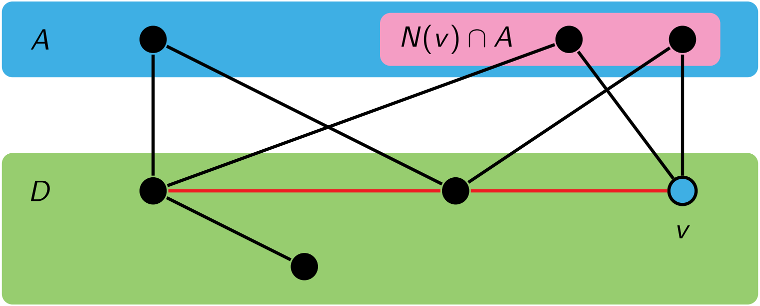

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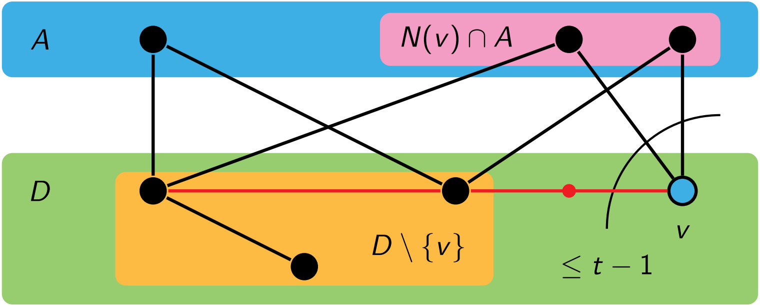

$$(\#\text{red edges incident to } v) + |N(v) \cap A| \leq t-1$$

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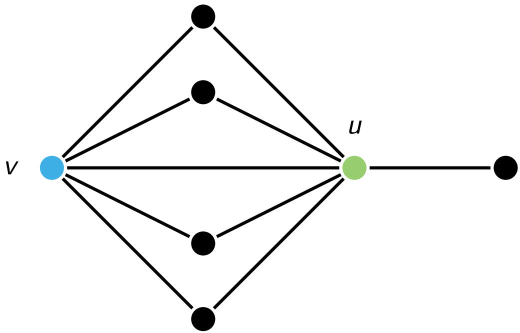

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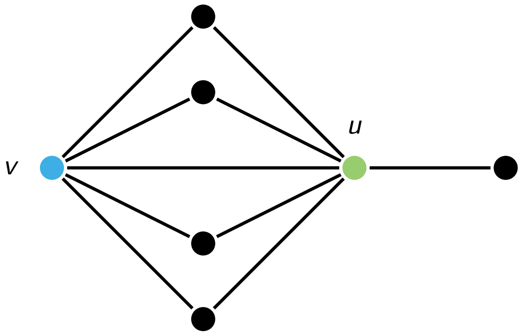

Then $u \in D_2$.

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