A parameterized approximation scheme for the 2D-Knapsack problem with wide items

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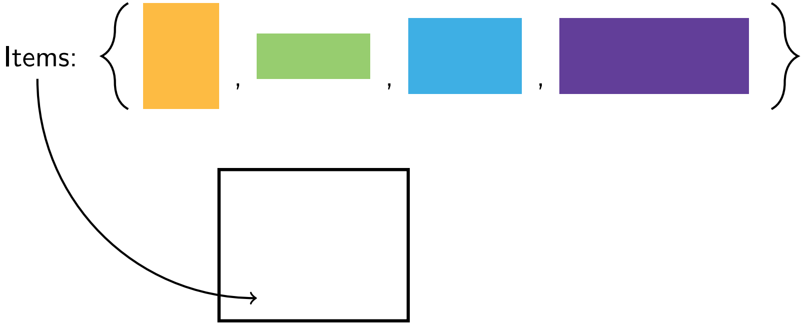
University of Warsaw

What is 2D-Knapsack?



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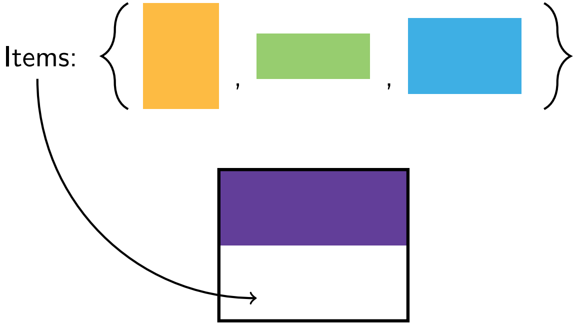

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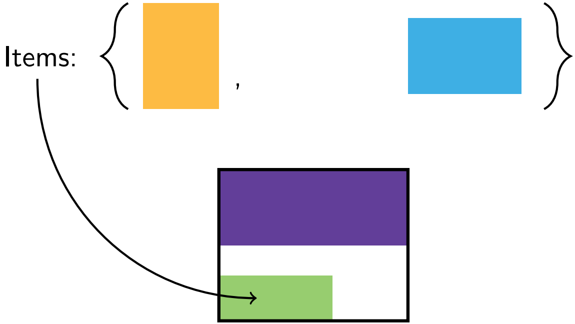

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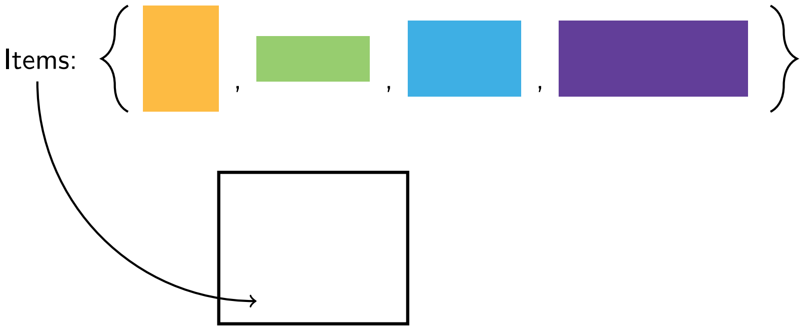

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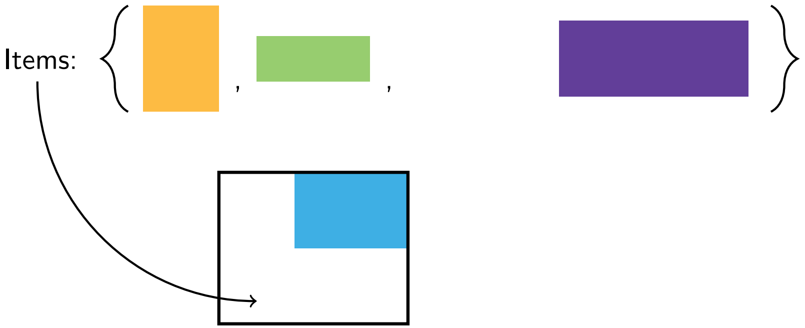

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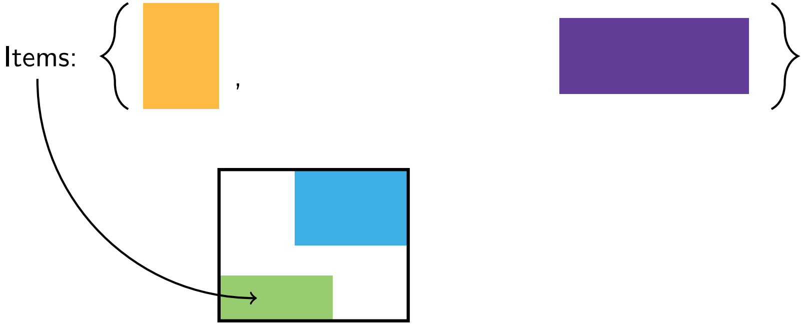
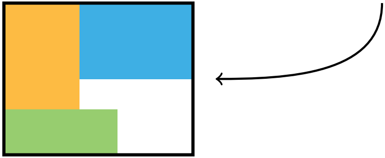


What is 2D-KNAPSACK?

Items:

✔ You've packed the maximum number of Items!



What is known about 2D-Knapsack?

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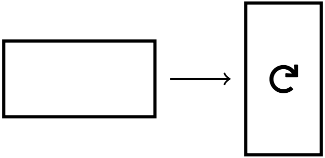

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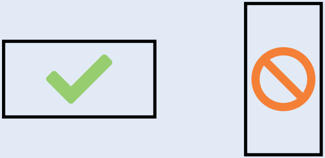

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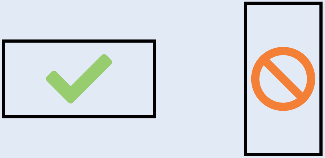

- *Box aspect ratio is bounded by δ*

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   $\implies$ keep only rectangles of substantial width
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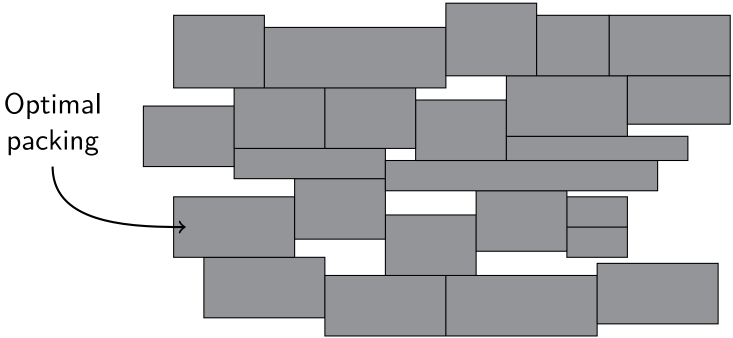
3. Dynamic Programming + color coding $\implies$ find the structured solution

The structural lemma

If all rectangles have substantial width, by deleting εk rectangles:



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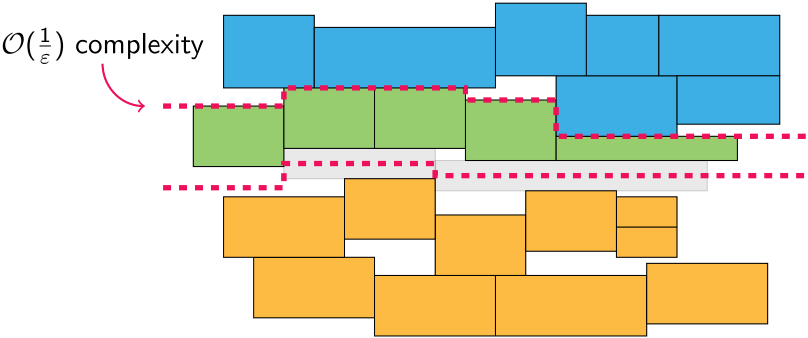

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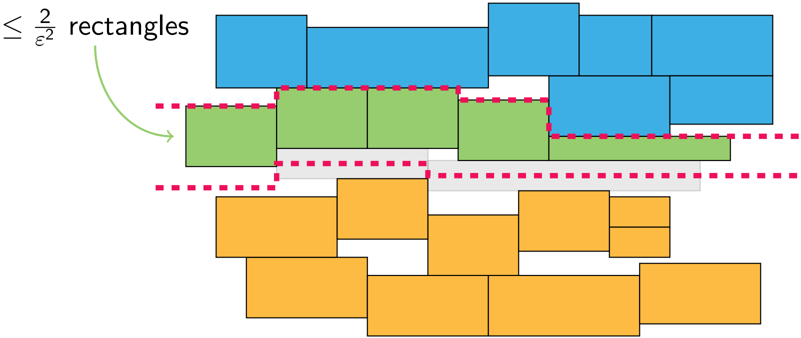

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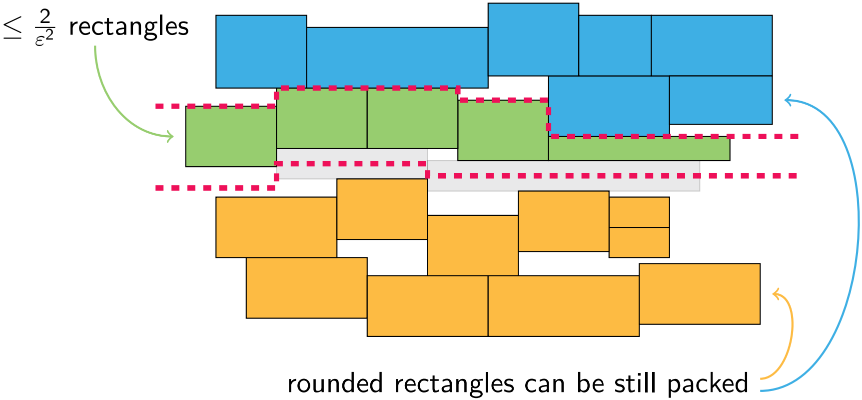

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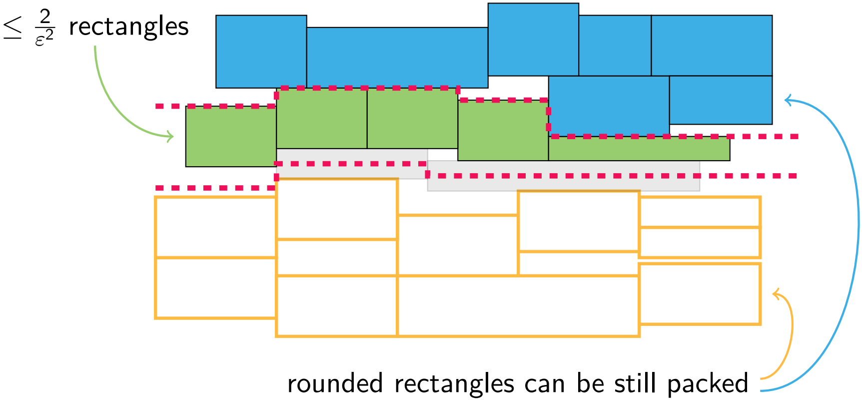

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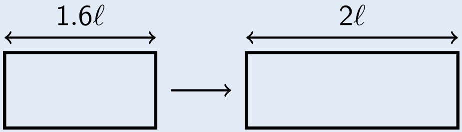

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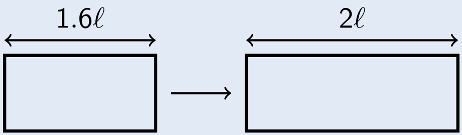

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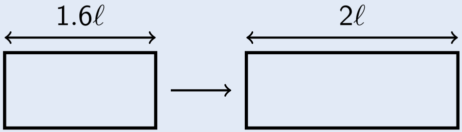

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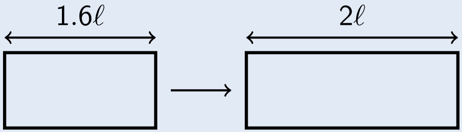

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3. Bounded amount of interesting rounded rectangles $\implies$ brute force.

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