A parameterized approximation scheme for the 2D-Knapsack problem with wide items

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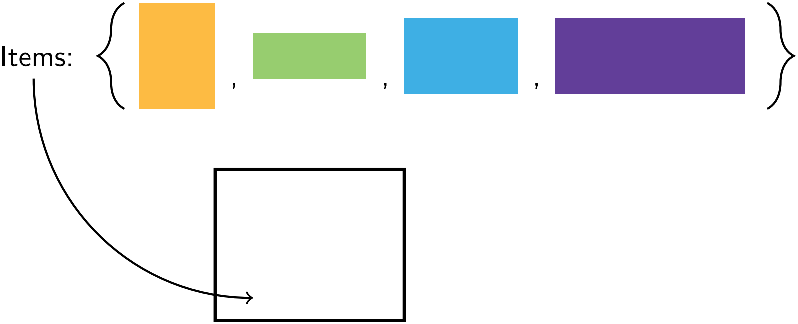
University of Warsaw

What is 2D-KNAPSACK?



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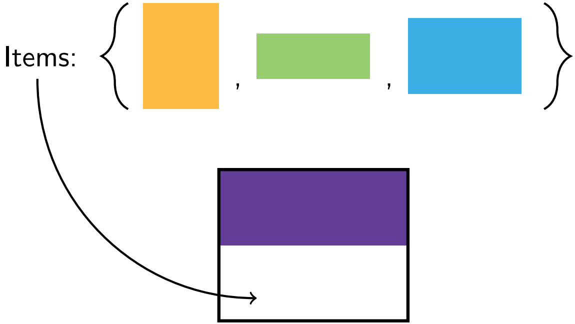

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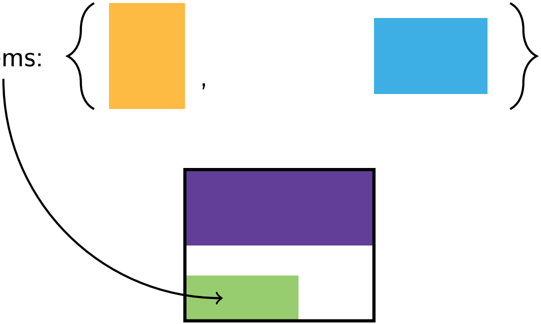

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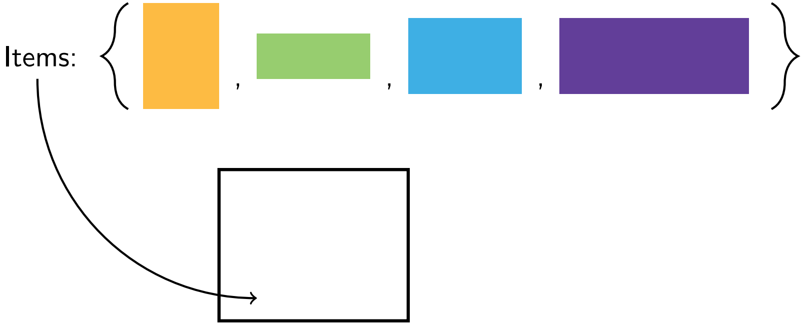

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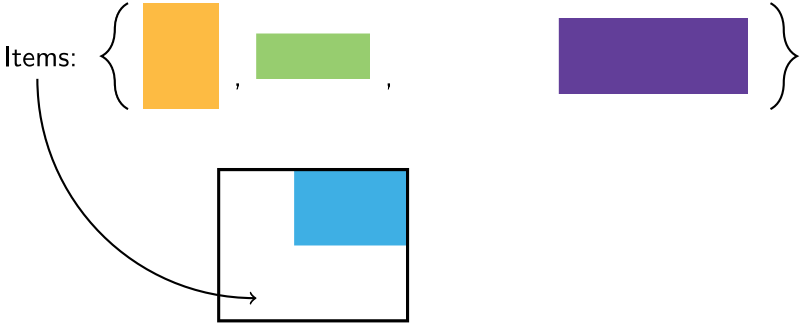

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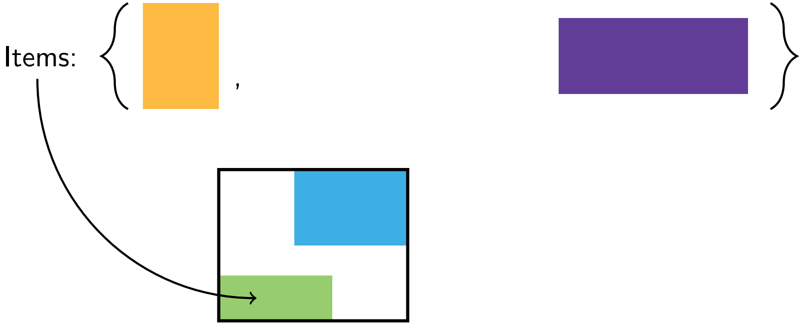
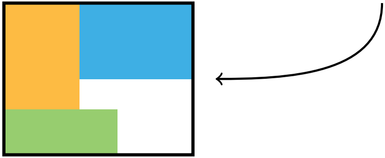


What is 2D-KNAPSACK?

Items: 

✔ You've packed the maximum number of Items for Amsterdam!



What is known about 2D-Knapsack?

[1] F. Grandoni, S. Kratsch, A. Wiese. "Parameterized Approximation Schemes for Independent Set of Rectangles and Geometric Knapsack". ESA 2019

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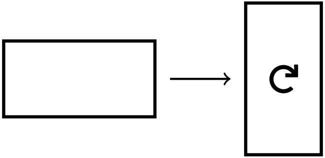

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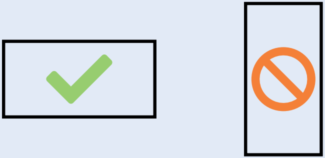

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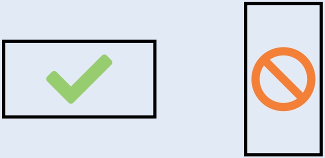

- *Box aspect ratio is bounded by δ*

Our approach

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   $\implies$ keep only rectangles of substantial width
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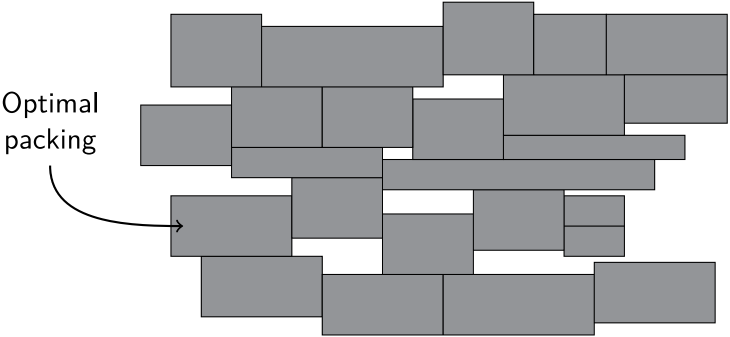
2. **Substantial width + remove εk rectangles**
   $\implies$ **WLOG, solution structured**
3. Dynamic Programming + color coding
   $\implies$ find the structured solution

The structural lemma

If all rectangles have substantial width, by deleting εk rectangles:



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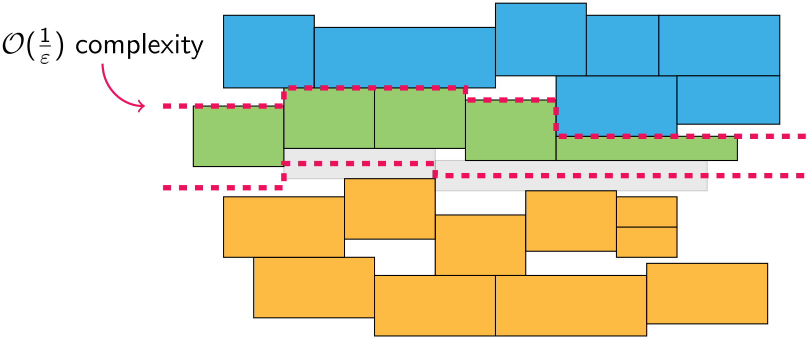

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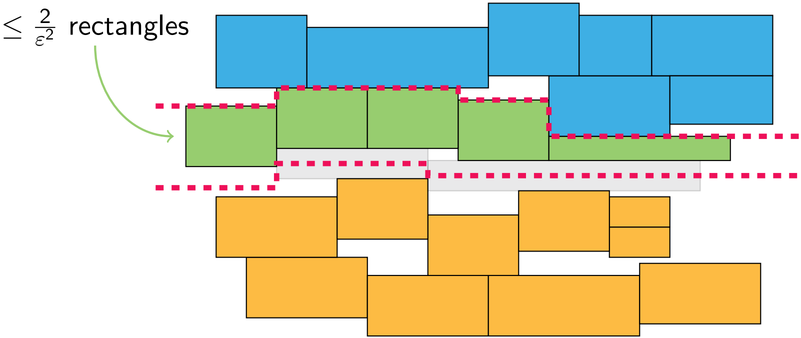

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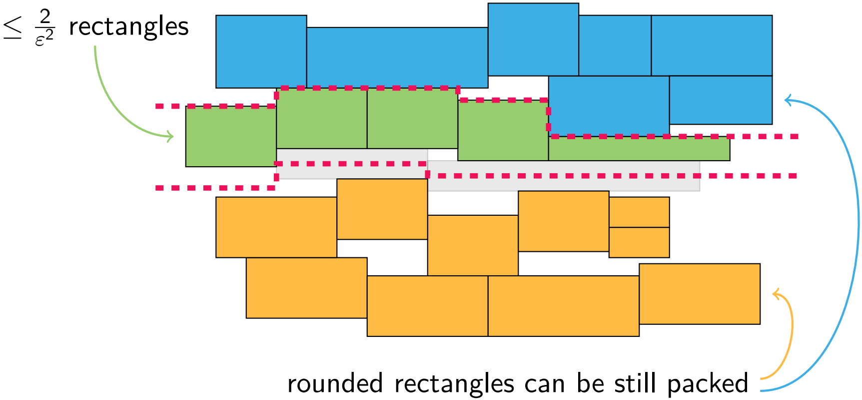

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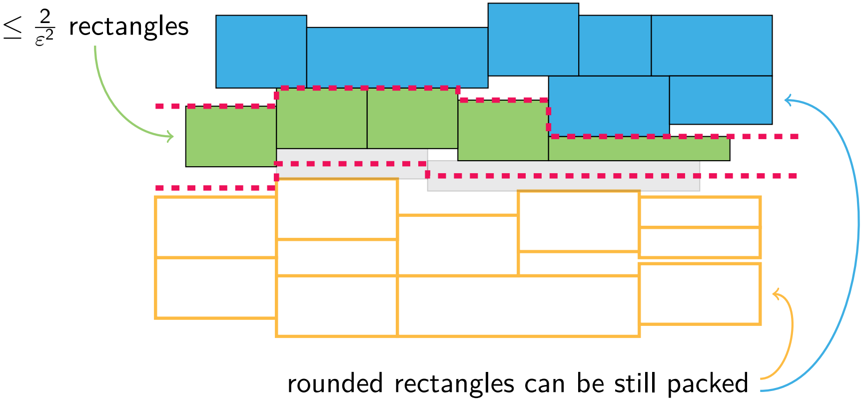

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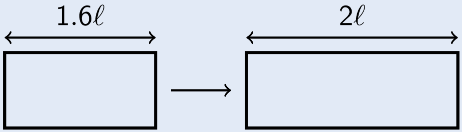

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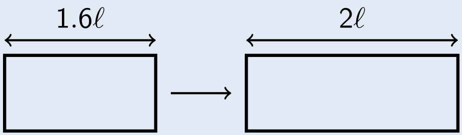

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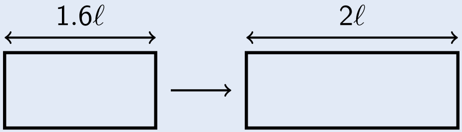

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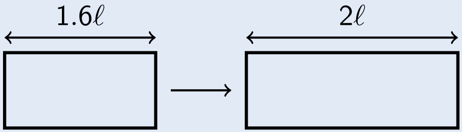

Properties

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2. Bounded number of possible widths: $\leq N_1/\ell = f(k,\delta)$.
3. Bounded amount of interesting rounded rectangles $\Longrightarrow$ brute force.

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- Open question:
  ? Find a PAS without the wide assumption
- Wide assumption necessary!
  ⚠ Guarantees reduction to rectangles of substantial width.

Thank You

Thanks for listening!