Induced Disjoint Paths Without an Induced Minor

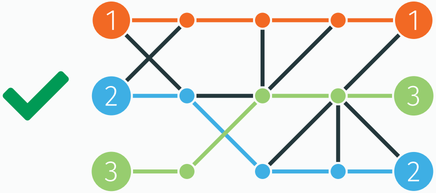
Pierre Aboulker [1] Édouard Bonnet [2] Timothé Picavet [3] Nicolas Trotignon [2]

[1]École normale Supérieure

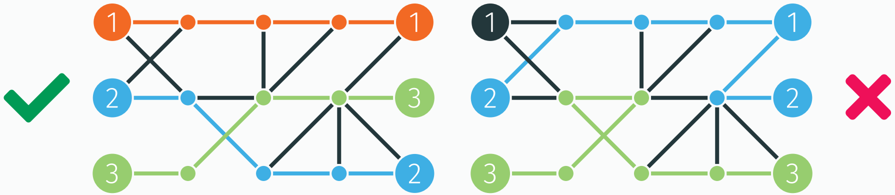
[2]ENS de Lyon

[3]LaBRI, U. Bordeaux

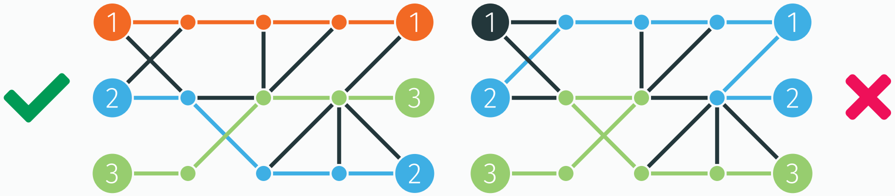
k-Disjoint Paths (polytime by Robertson and Seymour)



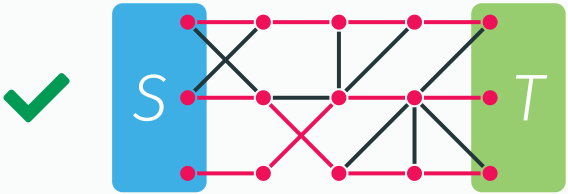
k-Disjoint Paths (polytime by Robertson and Seymour)



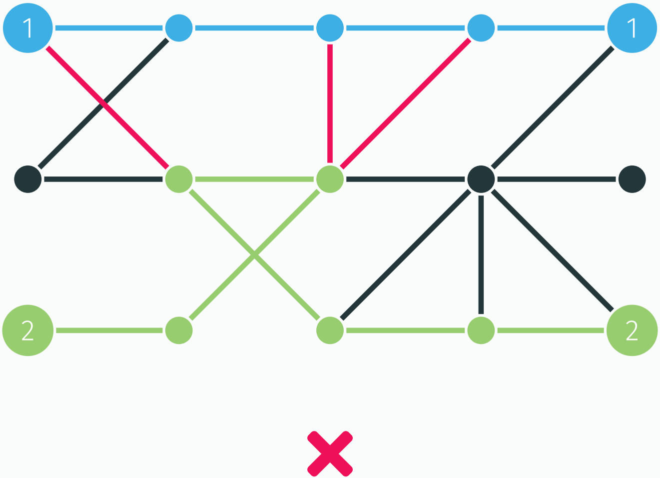
k-Disjoint Paths (polytime by Robertson and Seymour)



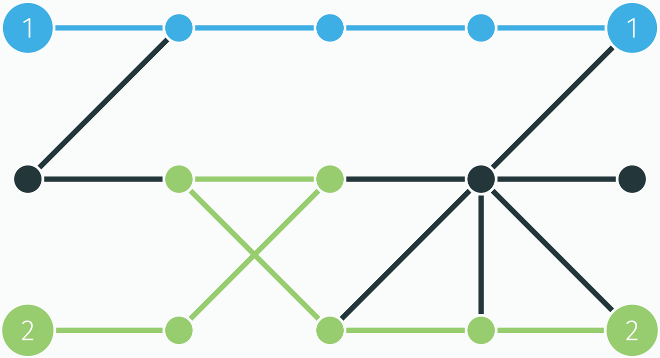
Disjoint S-T Paths (polytime by Menger's theorem)



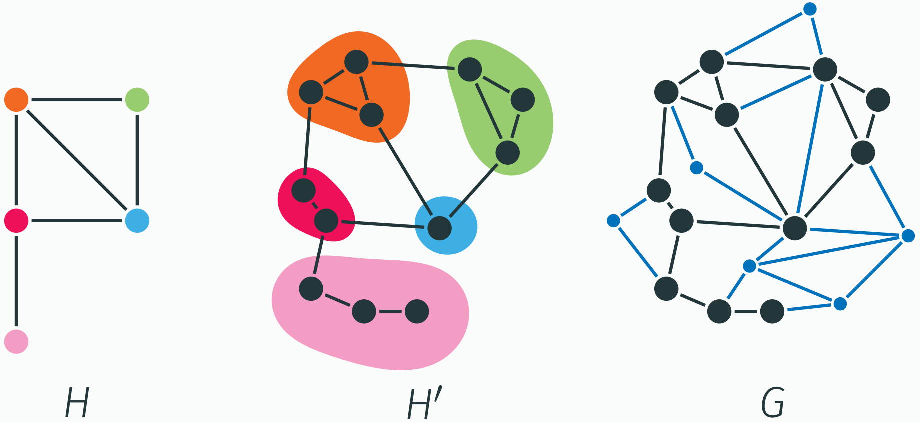
Induced k-Disjoint Paths



Induced k-Disjoint Paths

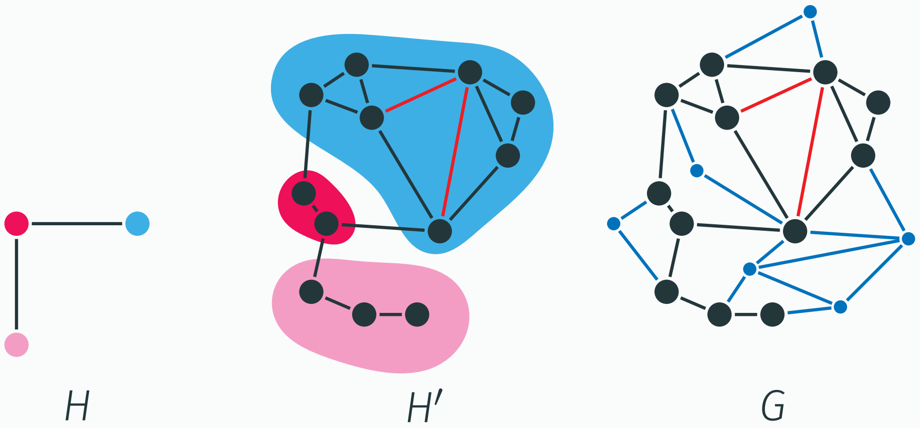


Graph minors



H is a minor of G

Induced graph minors



H is an *induced* minor of G

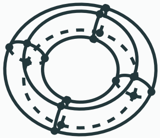
State of the art for Induced k-Disjoint Paths and Induced Disjoint S–T Paths

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

State of the art for INDUCED k-DISJOINT PATHS and INDUCED DISJOINT S–T PATHS

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:

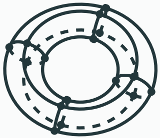


bounded genus
[Kobayashi, Kawarabayashi, 2009]

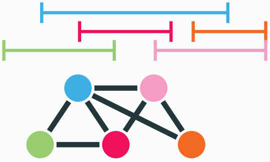
State of the art for Induced k-Disjoint Paths and Induced Disjoint S–T Paths

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:



bounded genus
[Kobayashi, Kawarabayashi, 2009]

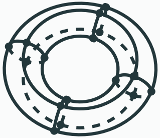


bounded + approx-imable mim-width
[Jaffke, Kwon, Telle, 2020]

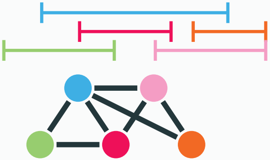
State of the art for INDUCED k-DISJOINT PATHS and INDUCED DISJOINT S–T PATHS

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:



bounded genus
[Kobayashi, Kawarabayashi, 2009]



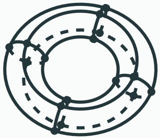
bounded + approx-imable mim-width
[Jaffke, Kwon, Telle, 2020]

claw-free
[Fiala, Kaminski, Lidický, Paulusma, 2012]

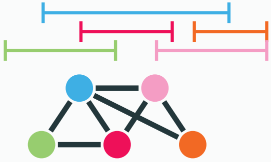
State of the art for INDUCED k-DISJOINT PATHS and INDUCED DISJOINT S–T PATHS

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:

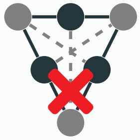


bounded genus
[Kobayashi, Kawarabayashi, 2009]



bounded + approx-imable mim-width
[Jaffke, Kwon, Telle, 2020]

claw-free
[Fiala, Kaminski, Lidický, Paulusma, 2012]

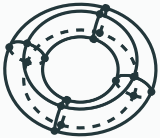


asteroidal-triple-free
[Golovach, Paulusma, van Leeuwen, 2022]

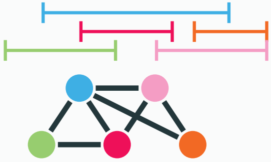
State of the art for INDUCED k-DISJOINT PATHS and INDUCED DISJOINT S–T PATHS

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:

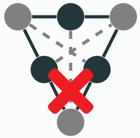


bounded genus
[Kobayashi, Kawarabayashi, 2009]

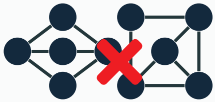


bounded + approximable mim-width
[Jaffke, Kwon, Telle, 2020]

claw-free
[Fiala, Kaminski, Lidický, Paulusma, 2012]



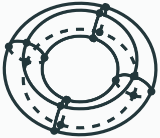
asteroidal-triple-free
[Golovach, Paulusma, van Leeuwen, 2022]



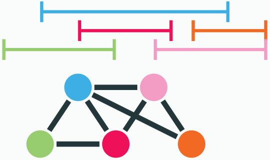
(theta,wheel)-free
[Radovanovic, Trotignon, Vuskovic, 2021]

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:

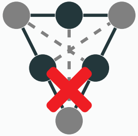


bounded genus
[Kobayashi, Kawarabayashi, 2009]

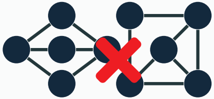


bounded + approximable mim-width
[Jaffke, Kwon, Telle, 2020]

claw-free
[Fiala, Kaminski, Lidický, Paulusma, 2012]



asteroidal-triple-free
[Golovach, Paulusma, van Leeuwen, 2022]



(theta,wheel)-free
[Radovanovic, Trotignon, Vuskovic, 2021]

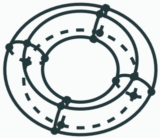
Question (Korhonen and Lokshtanov, 2024)

Is it NP-hard in H-induced-minor-free graphs for some fixed k and H?

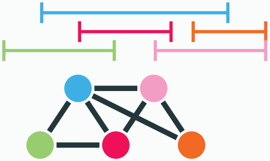
State of the art for Induced k-Disjoint Paths and Induced Disjoint S–T Paths

General graphs : NP-complete even for $k = 2$ / $|S| = |T| = 2$. [Fellows, 1989; Bienstock, 1992]

But polytime on:

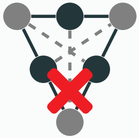


bounded genus
[Kobayashi, Kawarabayashi, 2009]

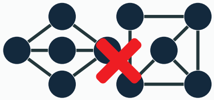


bounded + approximable mim-width
[Jaffke, Kwon, Telle, 2020]

claw-free
[Fiala, Kaminski, Lidický, Paulusma, 2012]



asteroidal-triple-free
[Golovach, Paulusma, van Leeuwen, 2022]



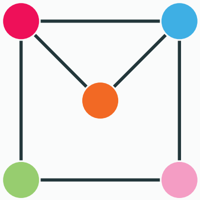
(theta,wheel)-free
[Radovanovic, Trotignon, Vuskovic, 2021]

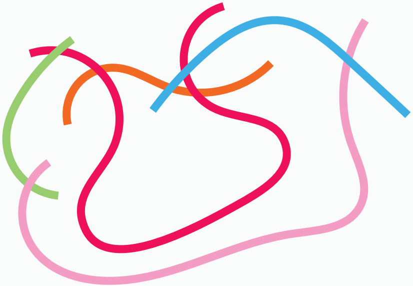
Question (Korhonen and Lokshtanov, 2024)

Is it NP-hard in H-induced-minor-free graphs for some fixed k and H?

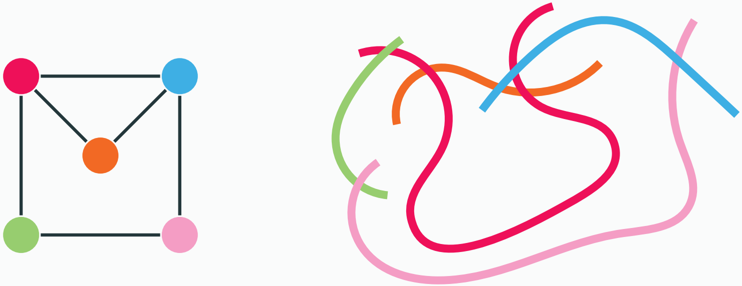
We solve this for $k = 2$ and H is the 1-subdivision of K_5 (or of $K_{3,3}$).

String graphs = intersection graphs of curves in the plane





String graphs = intersection graphs of curves in the plane



Observation

String graphs exclude the 1-subdivision of any non-planar graph as an induced minor.

Main theorem

Theorem (main)

Induced Disjoint S–T Paths with $|S| = |T| = 2$ is NP-complete in string graphs that are subgraphs of a constant power of bounded-degree planar graphs.

Main theorem

Theorem (main)

Induced Disjoint S–T Paths with $|S| = |T| = 2$ *is NP-complete in string graphs that are subgraphs of a constant power of bounded-degree planar graphs.*

Corollary

Induced 2-Disjoint Paths is hard on the same class.

Main theorem

Theorem (main)

Induced Disjoint S–T Paths with $|S| = |T| = 2$ is NP-complete in string graphs that are subgraphs of a constant power of bounded-degree planar graphs.

Corollary

Induced 2-Disjoint Paths is hard on the same class.

$\rightarrow$ Hard for H-induced-minor-free where H is the 1-subdivision of K_5.

Main theorem

Theorem (main)

Induced Disjoint S–T Paths with $|S| = |T| = 2$ is NP-complete in string graphs that are subgraphs of a constant power of bounded-degree planar graphs.

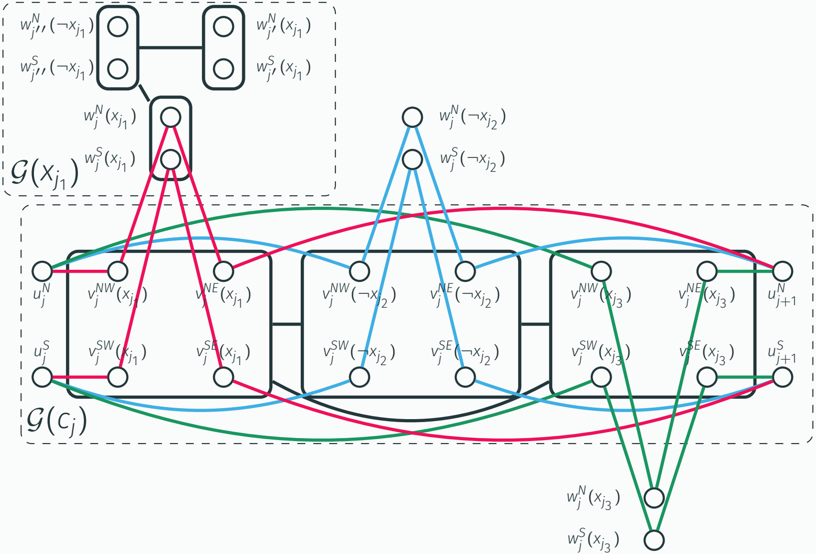
Corollary

Induced 2-Disjoint Paths is hard on the same class.

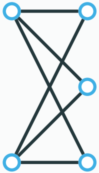
$\rightarrow$ Hard for H-induced-minor-free where H is the 1-subdivision of K_5.

$\rightarrow$ Requires time $2^{\Omega(\sqrt{n})}$ on string graphs of bounded maximum degree and twin-width, unless ETH fails.

The reduction: from E3-Occ E3-SAT

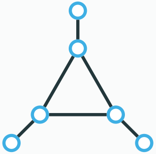


H-ISC is poly for $H =$



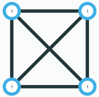
$K_{2,3}$

[Chudnovsky, Seymour, 2010]



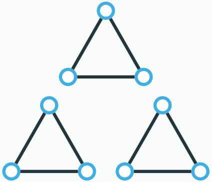
the net

[Chudnovsky, Seymour, Trotignon, 2013]



K_4

[Le, 2019]

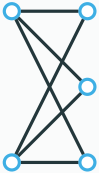


tK_3

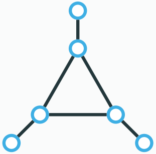
[Nguyen, Scott, Seymour, 2024]

Background: H-Induced Subdivision Containement

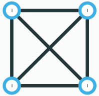
H-ISC is poly for $H =$



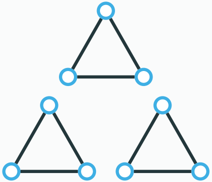
$K_{2,3}$
[Chudnovsky, Seymour, 2010]



the net
[Chudnovsky, Seymour, Trotignon, 2013]

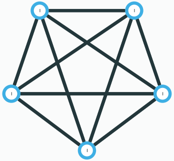


K_4
[Le, 2019]

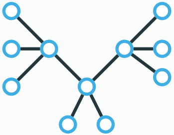


tK_3
[Nguyen, Scott, Seymour, 2024]

NP-hard for $H =$



K_5



Some trees

[Lévêque, Lin, Maffray, Trotignon, 2009]

Application: H-Induced Subdivision Containement

Question (Chudnovsky, Seymour, and Trotignon, 2013; Le, 2019)

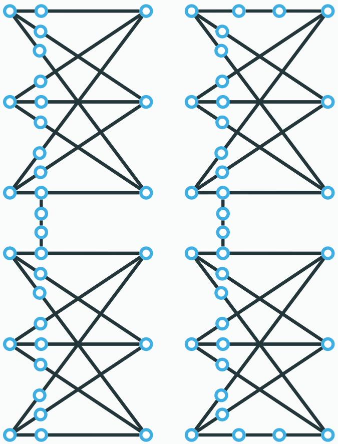
Is there is a polynomial-time algorithm for H-ISC for any subcubic graph H?

Application: *H*-Induced Subdivision Containement

Question (Chudnovsky, Seymour, and Trotignon, 2013; Le, 2019)

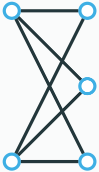
Is there is a polynomial-time algorithm for H-ISC for any subcubic graph H?

NO!



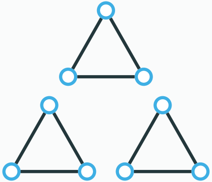
Background: H-Induced Minor Containement

H-IMC is poly for $H =$



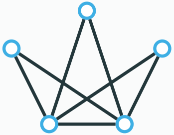
$K_{2,3}$

[Dallard, Dumas, Hilaire, Milanic, Perez, Trotignon, 2024]



tK_3

[Nguyen, Scott, Seymour, 2024]

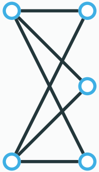


almost all 5-vertex graphs

see [Dallard, Dumas, Hilaire, Perez, 2025

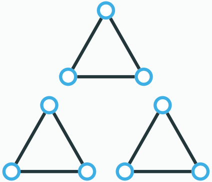
Background: H-Induced Minor Containement

H-IMC is poly for $H =$



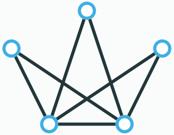
$K_{2,3}$

[Dallard, Dumas, Hilaire, Milanic, Perez, Trotignon, 2024]



tK_3

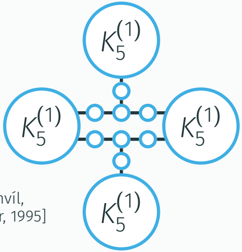
[Nguyen, Scott, Seymour, 2024]



almost all 5-vertex graphs

see [Dallard, Dumas, Hilaire, Perez, 2025

NP-hard for $H =$



[Fellows, Kratochvíl, Middendorf, Pfeiffer, 1995]

some tree with 2^{300} vertices

[Korhonen, Lokshtanov, 2024]

Application: H-Induced Minor Containement

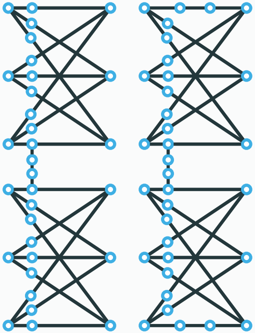
Question (Korhonen and Lokshtanov, 2024)

Polytime for a subcubic graph H with every edge incident to a vertex of degree 2?

Question (Korhonen and Lokshtanov, 2024)

Polytime for a subcubic graph H with every edge incident to a vertex of degree 2?

NO!

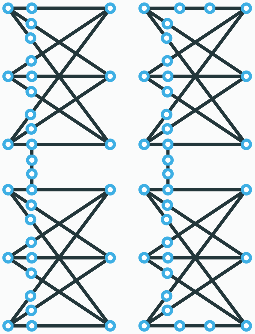


Application: H-Induced Minor Containement

Question (Korhonen and Lokshtanov, 2024)

Polytime for a subcubic graph H with every edge incident to a vertex of degree 2?

NO!



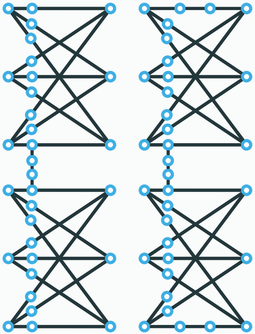
- $2^{\Omega(\sqrt{n})}$ unless ETH fails.

Application: H-Induced Minor Containement

Question (Korhonen and Lokshtanov, 2024)

Polytime for a subcubic graph H with every edge incident to a vertex of degree 2?

NO!



- $2^{\Omega(\sqrt{n})}$ unless ETH fails.
- $2^{\tilde{O}(n^{2/3})}$ by [Korhonen and Lokshtanov, 2024].

Conclusion: induced minor detection and induced disjoint paths is *hard* even for *simple graphs*!

Conclusion: induced minor detection and induced disjoint paths is *hard* even for *simple graphs*!

Some open questions:

- An hereditary class where Induced k-Disjoint Paths is NP-complete but Induced Disjoint S–T Paths is poly?

Conclusion: induced minor detection and induced disjoint paths is *hard* even for *simple graphs*!

Some open questions:

- An hereditary class where INDUCED k-DISJOINT PATHS is NP-complete but INDUCED DISJOINT S–T PATHS is poly?
- Restrict the graphs of the main theorem: 1-string graphs possible? (no pair of strings intersects twice)

Conclusion: induced minor detection and induced disjoint paths is *hard* even for *simple graphs*!

Some open questions:

- An hereditary class where INDUCED k-DISJOINT PATHS is NP-complete but INDUCED DISJOINT S–T PATHS is poly?
- Restrict the graphs of the main theorem: 1-string graphs possible? (no pair of strings intersects twice)

Conjecture

For any subcubic graph H, H-ISC is in P if and only if H is planar.

Conclusion: induced minor detection and induced disjoint paths is *hard* even for *simple graphs*!

Some open questions:

- An hereditary class where INDUCED k-DISJOINT PATHS is NP-complete but INDUCED DISJOINT S–T PATHS is poly?
- Restrict the graphs of the main theorem: 1-string graphs possible? (no pair of strings intersects twice)

Conjecture

For any subcubic graph H, H-ISC is in P if and only if H is planar.

☻ Thanks! ☻