Locally finding small dominating sets in $K_{2,t}$-minor-free graphs

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[1]LaBRI, Bordeaux

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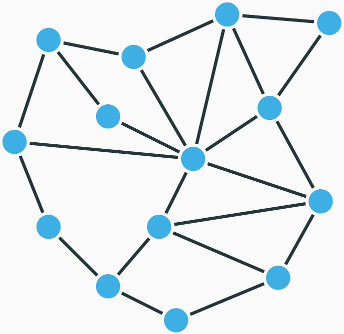
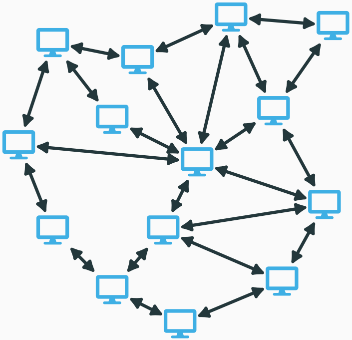
Centralized view



Distributed view



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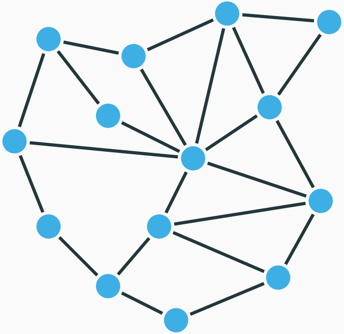
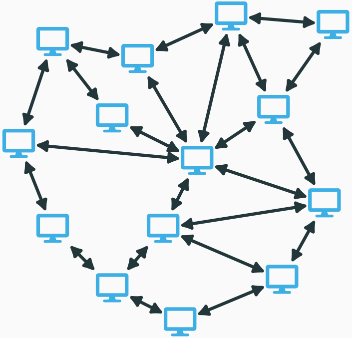


Focused on computing

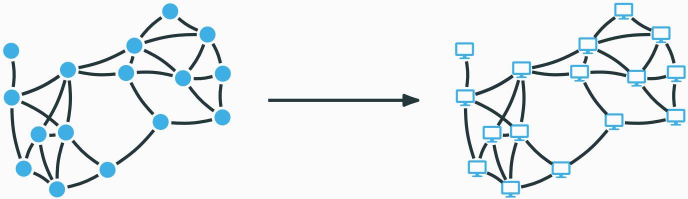
Distributed view



Focused on communication

The LOCAL model

graph



network

The LOCAL model

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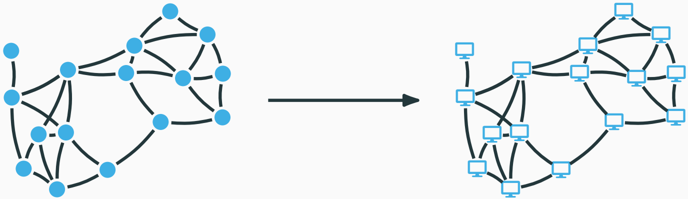


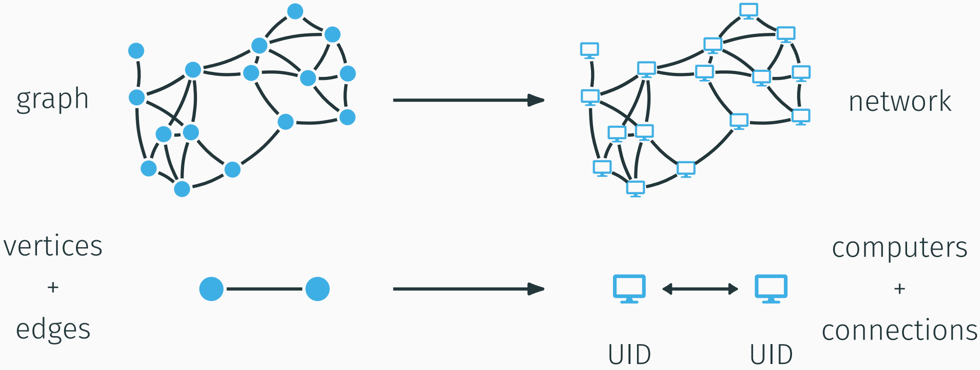
network

vertices
+
edges

computers
+
connections

UID UID

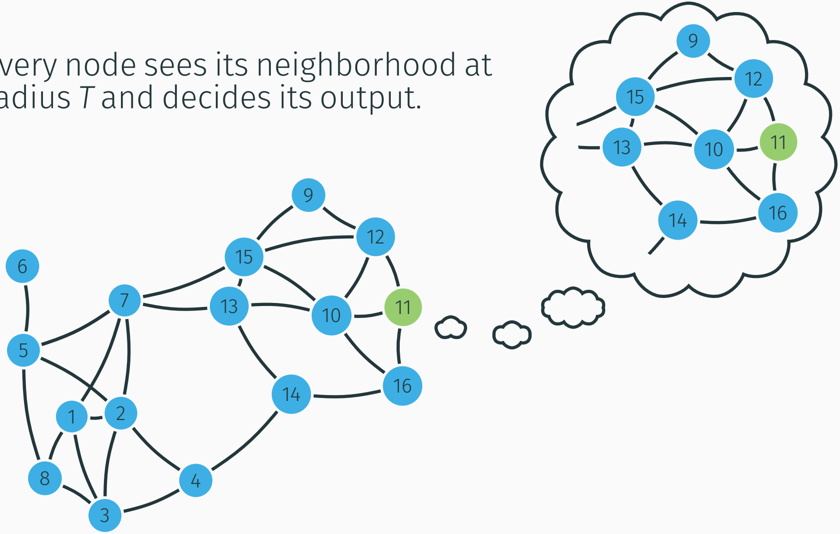
The LOCAL model



The network is also the input graph!

LOCAL running time T

Every node sees its neighborhood at radius T and decides its output.



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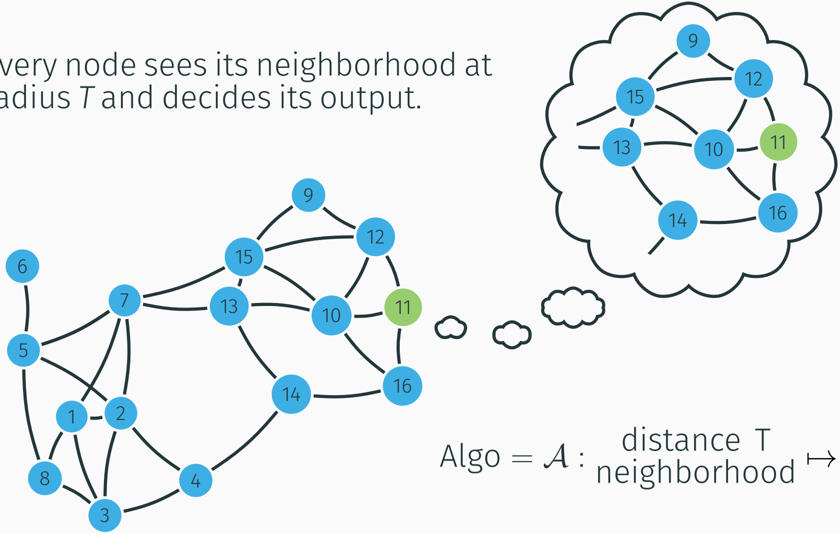
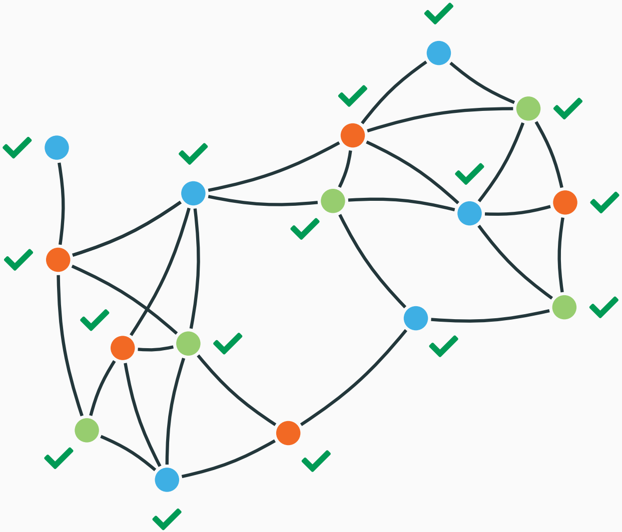


Algo $= \mathcal{A}$: distance T neighborhood $\mapsto$ local return value

An example: 3-coloring



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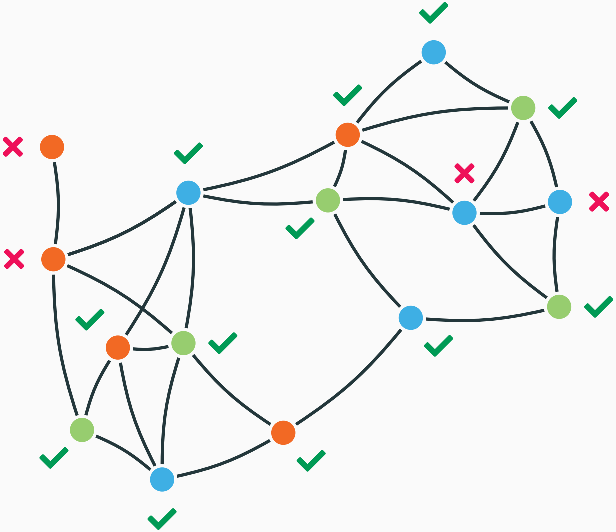
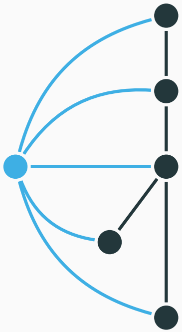


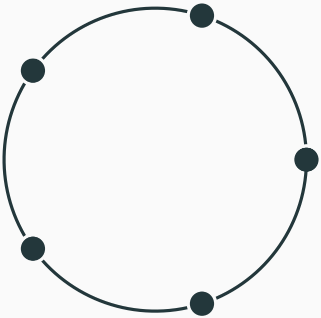
Complexity differences between LOCAL and centralized

Maximum Independent Set
when ∃ universal vertex



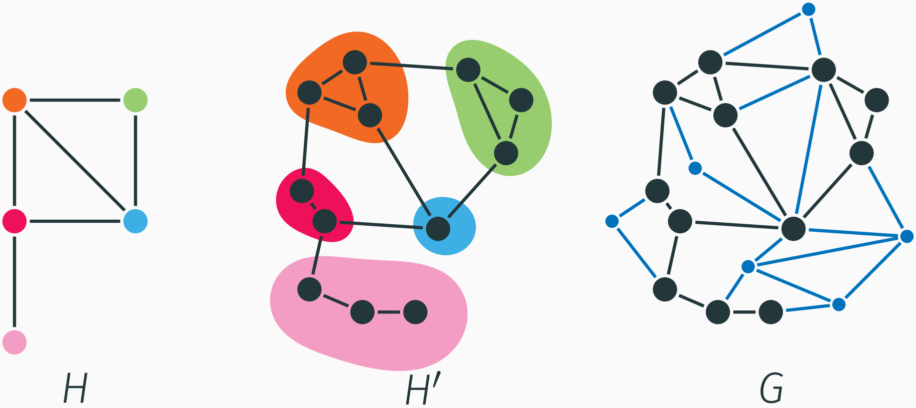
Easy in LOCAL
Hard in centralized

Detecting Cycles



Hard in LOCAL
Easy in centralized

Graph minors



H is a minor of G

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- $K_{2,t}$-minor-free graphs
  - $(2t - 1)$-approximation
  - Generalizes the outerplanar result

The algorithm

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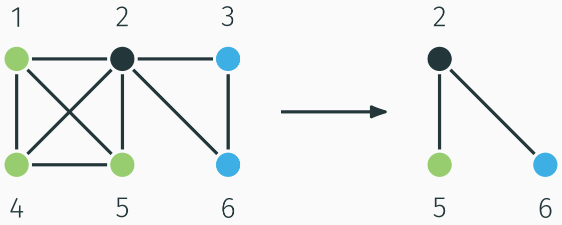
- Make G twinless (no vertices s.t. $N[u] = N[v]$)



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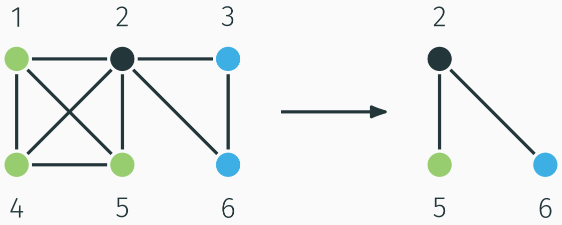
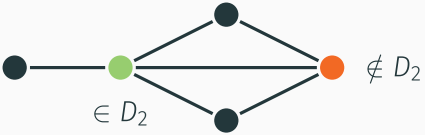


- Return $D_2 = \{v \in V(G) | \nexists u \in V(G - v), N[v] \subseteq N[u]\}$



Approximation factor

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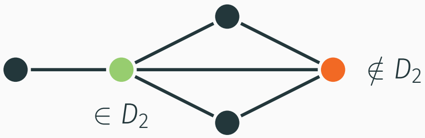


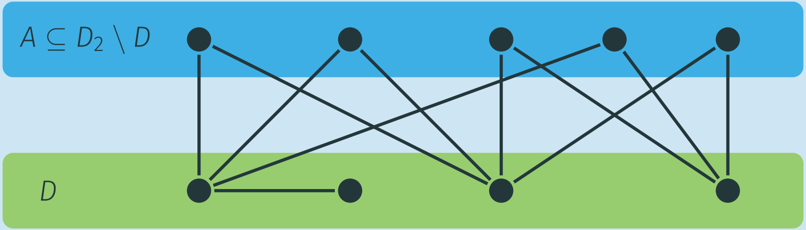
Theorem

Let D a MDS of G. If G is $K_{2,t}$-minor-free, then $|D_2| \leq (2t-1)|D|$.

Part 1: approximation factor

Lemma

Let D a MDS of G. Then $\exists H$ minor of G of the form:



with:

$$|A| \geq \frac{1}{2}|D_2 \setminus D|$$

$$\forall a \in A, |N(a) \cap D| \geq 2$$

Proof part 1: approximation factor

Lemma

Let $D = \{d_1, d_2, \ldots, d_k\}$ *a MDS of G. Then there exists H minor of G s.t.:*

- $V(H) = A \sqcup D$ *and* $A \subseteq D_2 \setminus D$
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Proof:

- Contract the branch sets $b_i = N[d_i] \setminus ((D_2 \setminus D) \cup \bigcup_{j<i} N[d_j] \cup \{d_{i+1}, \ldots, d_k\})$

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- For $v \in D_2 \setminus D$, $d_H(v) \geq 2$
- Contract some edges so that every vertex left in $D_2 \setminus D$ has 2 neighbors in D

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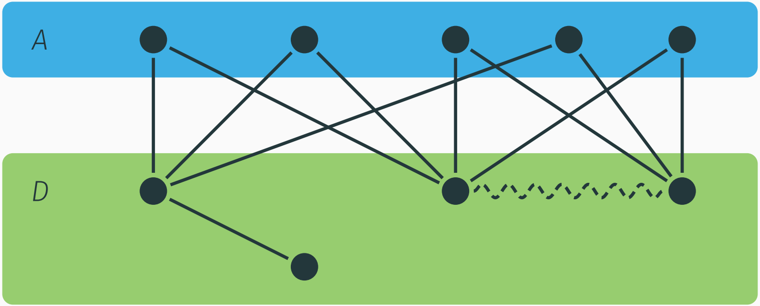

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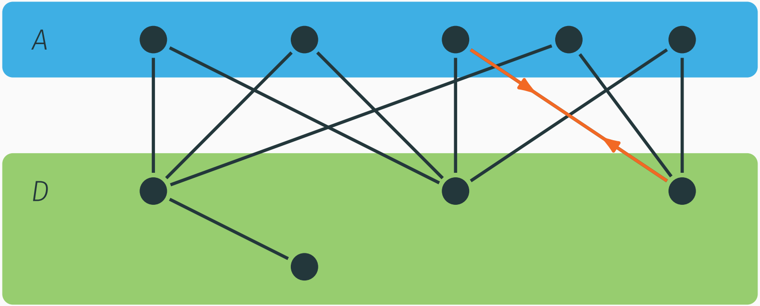

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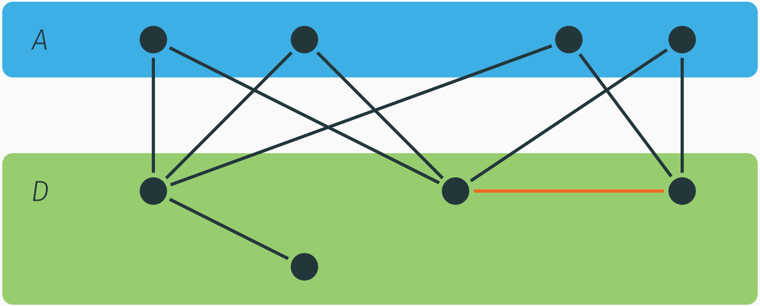

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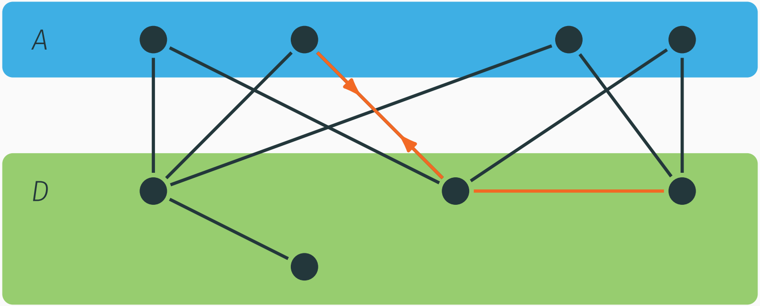

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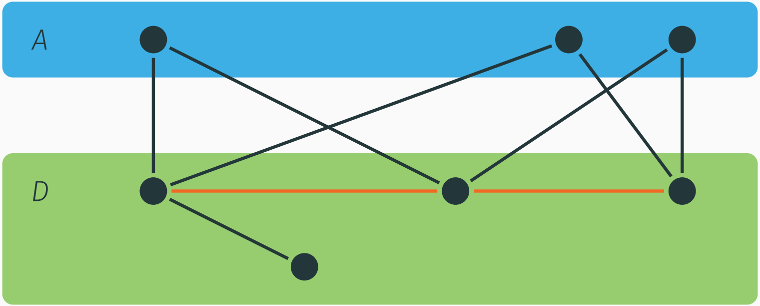

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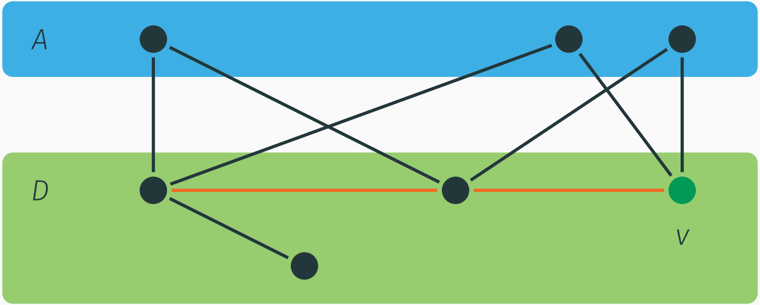

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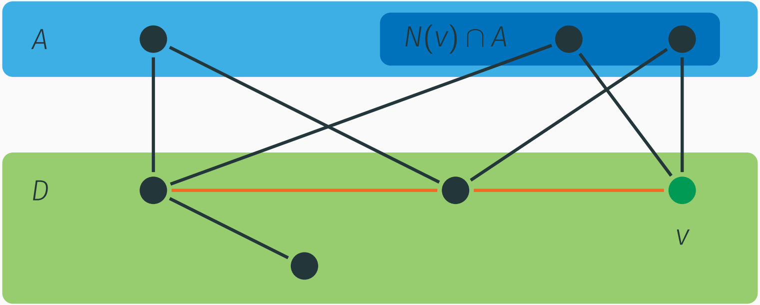

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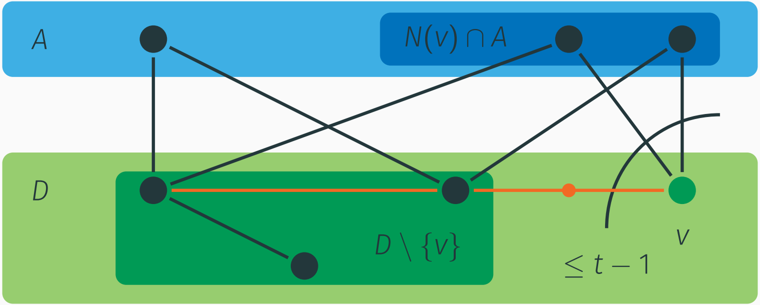
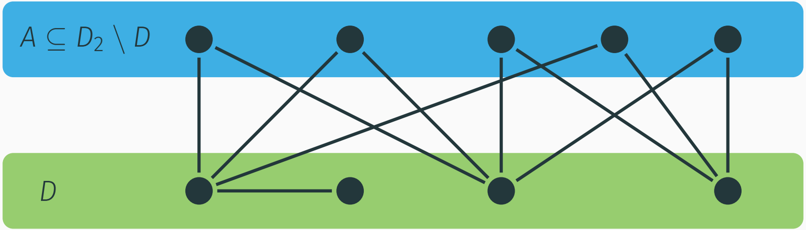


$$(\#\text{red edges incident to } v) + |N(v) \cap A| \leq t-1$$



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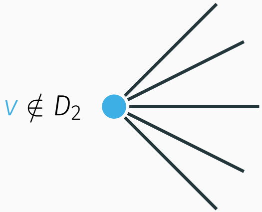

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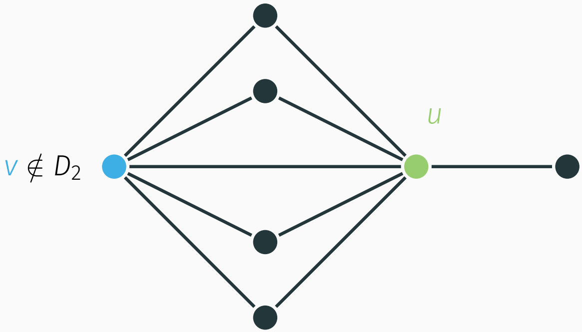

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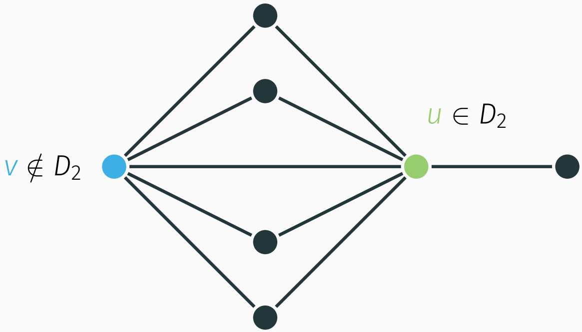

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